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An $N = 2$ worldsheet approach to D-branes in bihermitian geometries: I. Chiral and twisted chiral fields

Alexander Sevrin and Wieland Staessens*

Theoretische Natuurkunde, Vrije Universiteit Brussel and
The International Solvay Institutes
Pleinlaan 2, B-1050 Brussels, Belgium
E-mail: Alexandre.Sevrin@vub.ac.be, Wieland.Staessens@vub.ac.be

Alexander Wijns

Science Institute, University of Iceland,
Dunhaga 3, 107 Reykjavík, Iceland
E-mail: awijns@raunvis.hi.is

Abstract: We investigate $N = (2, 2)$ supersymmetric nonlinear σ-models in the presence of a boundary. We restrict our attention to the case where the bulk geometry is described by chiral and twisted chiral superfields corresponding to a bihermitian bulk geometry with two commuting complex structures. The D-brane configurations preserving an $N = 2$ worldsheet supersymmetry are identified. Duality transformations interchanging chiral for twisted chiral fields and vice versa while preserving all supersymmetries are explicitly constructed. We illustrate our results with various explicit examples such as the WZW-model on the Hopf surface $S^3 \times S^1$. The duality transformations provide e.g new examples of coisotropic A-branes on Kähler manifolds (which are not necessarily hyper-Kähler). Finally, by dualizing a chiral and a twisted chiral field to a semi-chiral multiplet, we initiate the study of D-branes in bihermitian geometries where the cokernel of the commutator of the complex structures is non-empty.

Keywords: D-branes, Superspaces, Sigma Models.

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Contents

1. Introduction

Non-linear σ-models in two dimensions with an $N = (2,2)$ supersymmetry, [1]–[4], are an important tool in the study of type II superstrings in the absence of R-R fluxes. The (local) target space geometry of such models is characterized by a metric, a closed 3-form and two complex structures. The complex structures are covariantly constant and the metric is hermitian with respect to both complex structures. These conditions can be solved in terms of a single real potential, [5] (building on results in [6]–[9]), which has a natural interpretation in $d = 2$, $N = (2,2)$ superspace where it is the Lagrange density. The Lagrange density is a function of three types of scalar superfields (satisfying certain constraints linear in the superspace derivatives) [5, 10]: chiral, twisted chiral and semi-chiral superfields.

Whenever one wants to deal with (open) strings propagating in backgrounds which include D-branes one necessarily needs to confront $N = (2,2)$ non-linear σ-models with boundaries. Having a boundary breaks the $N = (2,2)$ supersymmetry down to an $N = 2$ supersymmetry. While a lot of attention has been devoted to these models [11]–[17], their full description in $N = 2$ superspace remained till recently an unstudied problem. An initial investigation in [18] showed that this was straightforward as long as one only deals with chiral fields or put differently as long as one considers B-branes on Kähler manifolds. In [19] this was extended to A-branes on Kähler manifolds. The field content of these models consists exclusively of twisted chiral fields. The treatment of twisted chiral fields in $N = 2$ boundary superspace turned out to be rather subtle and an elegant and rich structure emerged. Duality transformations turning A- into B-branes and vice-versa were developed as well.

Kähler manifolds are only a particular example of the geometries which allow for an $N = (2,2)$ bulk supersymmetry. In general such a geometry is called bihermitian. In the present paper we extend the analysis of [19] to bihermitian geometries restricting ourselves to the simplest non-trivial case in which the two complex structures associated with the bihermitian geometry mutually commute. In $N = (2,2)$ superspace this corresponds to the case in which the bulk geometry is parameterized by chiral and twisted chiral superfields simultaneously. While still relatively simple, these models already encompass the Kähler case as they allow for non-trivial NS-NS backgrounds.

Finally, let us remark that the study of D-branes in the most general bihermitian geometry requires the introduction of semi-chiral $N = (2,2)$ superfields as well. This will appear elsewhere [20].

This paper is organized as follows. In the next section we briefly review supersymmetric non-linear σ-models in $N = 1$ boundary superspace. Section 3 introduces $N = 2$ boundary superspace together with the chiral and twisted chiral superfields. In section 4 we determine the boundary conditions which are allowed in the presence of chiral and twisted chiral superfields. The results of section 4 are illustrated by several explicit examples in section 5. The next section discusses duality transformations which interchange chiral for twisted chiral fields and vice-versa. In addition we also briefly discuss the duality between a pair consisting of a chiral and a twisted chiral superfield and a semi-chiral multiplet. We end with conclusions and an outlook. Our conventions are summarized in the appendix.

JHEP10(2008)108

2. From $N = (1,1)$ to $N = 1$

In the absence of boundaries a non-linear σ-model (with $N \leq (1,1)$) on some d-dimensional target manifold $\mathcal{M}$ is characterized by a metric $g_{ab}(X)$ and a closed 3-form $T_{abc}(X)$ (the latter is known as the torsion, the Kalb-Ramond 3-form or the NS-NS form) on $\mathcal{M}$ where X^a are local coordinates on $\mathcal{M}$ and $a, b, c, \dots \in \{1, \dots, d\}$. The action in $N = (1,1)$ superspace is simply,[1]

$$\mathcal{S} = 8 \int d^2\sigma\, d^2\theta\, D_+ X^a D_- X^b \left(g_{ab} + b_{ab}\right), \tag{2.1}$$

where we used a locally defined 2-form potential $b_{ab}(X) = -b_{ba}(X)$ for the torsion,

$$T_{abc} = -\frac{3}{2}\, \partial_{[a} b_{bc]}. \tag{2.2}$$

We consider a boundary at $\sigma = 0$ ($\sigma \geq 0$) and $\theta^+ = \theta^-$. This breaks the invariance under translations in both the σ and the $\theta' \equiv \theta^+ - \theta^-$ direction thus reducing the $N = (1,1)$ supersymmetry to an $N = 1$ supersymmetry. We introduce the derivatives,

$$D \equiv D_+ + D_-, \qquad D' \equiv D_+ - D_-, \tag{2.3}$$

which satisfy,

$$D^2 = D'^2 = -\frac{i}{2}\partial_\tau, \qquad \{D, D'\} = -i\, \partial_\sigma, \tag{2.4}$$

and,

$$D_+ D_- = -\frac{1}{2}\, D\, D' - \frac{i}{4}\, \partial_\sigma. \tag{2.5}$$

The action,

$$\mathcal{S} = -4 \int d^2\sigma\, d\theta\, D' \Big(D_+ X^a D_- X^b \left(g_{ab} + b_{ab}\right) \Big), \tag{2.6}$$

is manifestly invariant under the surviving $N = 1$ supersymmetry and — because of eq. (2.5) – differs from the action in eq. (2.1) by a boundary term [21, 18]. Upon performing the D' derivative one gets the action in $N = 1$ boundary superspace previously obtained in [18],

$$\begin{aligned}\mathcal{S} = \int d^2\sigma\, d\theta \Big(& i\, g_{ab}\, D X^a \partial_\tau X^b - 2i\, g_{ab}\, \partial_\sigma X^a D' X^b + 2i\, b_{ab}\, \partial_\sigma X^a D X^b \\ & -2\, g_{ab}\, D' X^a \nabla D' X^b + 2\, T_{abc}\, D' X^a D X^b D X^c - \frac{2}{3}\, T_{abc}\, D' X^a D' X^b D' X^c \Big),\end{aligned} \tag{2.7}$$

where,

$$\nabla D' X^a \equiv D D' X^a + \left\{ {a \atop bc} \right\} D X^b D' X^c. \tag{2.8}$$

[1] Our conventions are given in appendix A. Note that we have rescaled the scalar fields with a factor $\sqrt{2\pi\alpha'}$ in order to make them dimensionless.

Both X^a and $D'X^a$ are now independent $N = 1$ superfields. Adding a boundary term $\mathcal{S}_b$ to the action eq. (2.7),

$$\mathcal{S}_b = 2i \int d\tau \, d\theta \, A_a \, DX^a, \tag{2.9}$$

is equivalent to the modification $b_{ab} \to \mathcal{F}_{ab} = b_{ab} + F_{ab}$ with $F_{ab} = \partial_a A_b - \partial_b A_a$ in eq. (2.7).

Varying the action eq. (2.6)[2] or eq. (2.7) yields a boundary term,

$$\delta \mathcal{S}\big|_{\text{boundary}} = -2i \int d\tau d\theta \, \delta X^a \left(g_{ab} \, D'X^b - b_{ab} \, DX^b \right), \tag{2.10}$$

which will only vanish upon imposing suitable boundary conditions. In order to do this we introduce an almost product structure, a (1,1) tensor $R(X)^a{}_b$ [11, 15, 16, 18], which satisfies,

$$R^a{}_c \, R^c{}_b = \delta^a_b, \tag{2.11}$$

and projection operators $\mathcal{P}_\pm$,

$$\mathcal{P}^a_{\pm b} \equiv \frac{1}{2} \left(\delta^a_b \pm R^a{}_b \right). \tag{2.12}$$

With this we impose Dirichlet boundary conditions,

$$\mathcal{P}^a_{-b} \, \delta X^b = 0. \tag{2.13}$$

Using eq. (2.13), one verifies that the boundary term eq. (2.10) vanishes, provided one imposes in addition the Neumann boundary conditions,

$$\mathcal{P}_{+ba} \, D'X^b = \mathcal{P}^b_{+a} \, b_{bc} \, DX^c. \tag{2.14}$$

If in addition we have that $R_{ab} = R_{ba}$ with $R_{ab} = g_{ac} R^c{}_b$, then we can rewrite the Neumann boundary conditions as,

$$\mathcal{P}^a_{+b} D'X^b = \mathcal{P}^a_{+c} b^c{}_d \mathcal{P}^d_{+b} DX^b, \tag{2.15}$$

and $\mathcal{P}_+$ and $\mathcal{P}_-$ resp. project onto Neumann and Dirichlet directions resp. Note that as was discussed in [19] this is not necessary.

Invariance of the Dirichlet boundary conditions under what remains of the super-Poincaré transformations implies that on the boundary,

$$\mathcal{P}^a_{-b} \, DX^b = \mathcal{P}^a_{-b} \, \partial_\tau X^b = 0, \tag{2.16}$$

hold as well. Using $D^2 = -i/2 \, \partial_\tau$, we get from eq. (2.16) the integrability conditions,[3]

$$0 = \mathcal{P}^d_{+[b} \mathcal{P}^e_{+c]} \mathcal{P}^a_{+d,e} = -\frac{1}{2} \mathcal{P}^a_{-e} \mathcal{N}^e{}_{bc}[R, R]. \tag{2.17}$$

[2] Here one uses that $\int d^2\sigma d\theta D' \, D_\pm = -(i/2) \int d\tau d\theta$.

[3] Out of two $(1,1)$ tensors $R^a{}_b$ and $S^a{}_b$, one constructs a $(1,2)$ tensor $\mathcal{N}[R,S]^a{}_{bc}$, the Nijenhuis tensor, as $\mathcal{N}[R,S]^a{}_{bc} = R^a{}_d S^d{}_{[b,c]} + R^d{}_{[b} S^a{}_{c],d} + R \leftrightarrow S$.

These conditions guarantee the existence of adapted coordinates $X^{\hat{a}}$, $\hat{a} \in \{p+1,\ldots,d\}$, with $p \leq d$ the rank of $\mathcal{P}_+$ such that the Dirichlet boundary conditions, eq. (2.13) are simply given by,

$$X^{\hat{a}} = \text{ constant}, \qquad \forall\, \hat{a} \in \{p+1,\ldots,d\}. \tag{2.18}$$

Writing the remainder of the coordinates as $X^{\check{a}}$, $\check{a} \in \{1,\ldots,p\}$, we get the Neumann boundary conditions, eq. (2.14), in our adapted coordinates,

$$g_{\check{a}b}\, D' X^b = b_{\check{a}\check{b}}\, DX^{\check{b}}, \tag{2.19}$$

where b is summed from 1 to d and we used that $DX^{\hat{b}}$ vanishes on the boundary. Concluding, the action eq. (2.6) together with the boundary conditions eqs. (2.18) and (2.19), describe open strings in the presence of a Dp-brane whose position is determined by eq. (2.18).

3. N=2 superspace

3.1 $N = (2,2)$ supersymmetry in the absence of boundaries

Already in the absence of boundaries, promoting the $N = (1,1)$ supersymmetry of the action in eq. (2.1) to an $N = (2,2)$ supersymmetry introduces additional structure. The most general extra supersymmetry transformations — consistent with dimensions and super Poincaré symmetry — are of the form,

$$\delta X^a = \varepsilon^+\, J_+^a{}_b(X)\, D_+ X^b + \varepsilon^-\, J_-^a{}_b(X)\, D_- X^b, \tag{3.1}$$

which implies the introduction of two (1,1) tensors J_+ and J_-. On-shell closure of the algebra requires both J_+ and J_- to be complex structures,

$$\begin{aligned} J_{\pm}^a{}_c\, J_{\pm}^c{}_b &= -\delta^a_b, \\ N[J_\pm, J_\pm]^a{}_{bc} &= 0, \end{aligned} \tag{3.2}$$

while invariance of the action necessitates that the metric is hermitian with respect to *both* complex structures,[4]

$$J_{\pm}^c{}_a\, J_{\pm}^d{}_b\, g_{cd} = g_{ab}\,, \tag{3.3}$$

and that both complex structures have to be covariantly constant,

$$0 = \nabla_c^{\pm}\, J_{\pm}^a{}_b \equiv \partial_c\, J_{\pm}^a{}_b + \Gamma_{\pm}^a{}_{dc} J_{\pm}^d{}_b - \Gamma_{\pm}^d{}_{bc} J_{\pm}^a{}_d\,, \tag{3.4}$$

with the connections $\Gamma_\pm$ given by,

$$\Gamma_{\pm}^a{}_{bc} \equiv \left\{{a \atop bc}\right\} \pm T^a{}_{bc}\,. \tag{3.5}$$

[4]This implies the existence of two two-forms $\omega^\pm_{ab} = -\omega^\pm_{ba} = g_{ac} J_{\pm}^c{}_b$. In general they are not closed. Using eq. (3.4), one shows that $\omega^\pm_{[ab,c]} = \mp 2 J_{\pm}^d{}_{[a} T_{bc]d} = \mp(2/3) J_{\pm}^d{}_a J_{\pm}^e{}_b J_{\pm}^f{}_c T_{def}$, where for the last step we used the fact that the Nijenhuis tensors vanish.

The targetmanifold geometry $(\mathcal{M}, g, J_\pm, T)$ consists of a bihermitian manifold (the manifold has two complex structures for both of which the metric is hermitian) and both complex structures are covariantly constant with respect to different connections which are given in eq. (3.5). When the torsion vanishes, this type of geometry reduces to the usual Kähler geometry.

An interesting observation is that all terms in the algebra which do not close off-shell are proportional to the commutator of the complex structures $[J_+, J_-]$ suggesting that extra auxiliary fields will be needed in the direction of coker$[J_+, J_-]$. A detailed analysis revealed the following picture (suggested in [8, 9] and [7] and shown to be correct in [5]): writing $\ker[J_+, J_-] = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-)$, one gets that $\ker(J_+ - J_-)$ and $\ker(J_+ + J_-)$ resp. can be integrated to chiral and twisted chiral multiplets resp. [2]. Semi-chiral multiplets [6] are required for the description of coker$[J_+, J_-]$. The Lagrange density is a real function of these superfields. Metric, torsion and the complex structures can all be expressed in terms of this function. When only chiral and twisted chiral fields are present the relations are all linear while once semi-chiral fields are present as well non-linearities appear. This clearly shows that this geometry generalizes Kähler geometry: the whole local geometry is encoded in a single real function which generalizes the Kähler potential. As a consequence such geometries are often called generalized Kähler geometries.[5]

In the present paper we will focus on chiral and twisted chiral multiplets, i.e. we assume that J_+ and J_- commute.[6] These fields in $N = (2, 2)$ superspace (once more we refer to the appendix for conventions) satisfy the constraints $\hat{D}_\pm X^a = J_{\pm}{}^a{}_b \, D_\pm X^b$ where J_+ and J_- can be simultaneously diagonalized. When the eigenvalues of J_+ and J_- have the same (the opposite) sign we have chiral (twisted chiral) superfields. Explicitly, we get that chiral superfields X^α, $\alpha \in \{1, \dots, m\}$, satisfy,

$$\hat{D}_\pm X^\alpha = +i \, D_\pm X^\alpha, \qquad \hat{D}_\pm X^{\bar{\alpha}} = -i \, D_\pm X^{\bar{\alpha}}. \tag{3.6}$$

Twisted chiral superfields X^μ, $\mu \in \{1, \dots, n\}$ satisfy,

$$\hat{D}_\pm X^\mu = \pm i \, D_\pm X^\mu, \qquad \hat{D}_\pm X^{\bar{\mu}} = \mp i \, D_\pm X^{\bar{\mu}}. \tag{3.7}$$

The most general action involving these superfields is given by,

$$\mathcal{S} = 4 \int d^2\sigma \, d^2\theta \, d^2\hat{\theta} \, V(X, \bar{X}), \tag{3.8}$$

where the Lagrange density $V(X, \bar{X})$ is an arbitrary real function of the chiral and twisted chiral superfields. It is only defined modulo a generalized Kähler transformation,

$$V \to V + F + \bar{F} + G + \bar{G}, \tag{3.9}$$

with,

$$\partial_{\bar{\alpha}} F = \partial_{\bar{\mu}} F = 0, \qquad \partial_{\bar{\alpha}} G = \partial_\mu G = 0. \tag{3.10}$$

[5] This can be made very concrete in the framework of Hitchin's generalized complex geometry, see e.g. [22 – 24] and references therein.

[6] As already mentioned in the introduction we relegate the study of the most general case — which includes the semi-chiral superfields — to a forthcoming paper [20].

Passing to $N = (1,1)$ superspace and comparing the result to eq. (2.1), allows one to identify the metric and the torsion potential,[7]

$$\begin{aligned} g_{\alpha\bar{\beta}} &= +V_{\alpha\bar{\beta}}, \qquad g_{\mu\bar{\nu}} = -V_{\mu\bar{\nu}}, \\ b_{\alpha\bar{\nu}} &= -V_{\alpha\bar{\nu}}, \qquad b_{\mu\bar{\beta}} = +V_{\mu\bar{\beta}}, \end{aligned} \tag{3.11}$$

where all other components of g and b vanish. When writing $V_{\alpha\bar{\beta}}$, we mean $\partial_\alpha\partial_{\bar{\beta}}V$ etc. Let us end with two remarks. Interchanging the chiral with the twisted chiral superfields and vice-versa while sending $V \to -V$, leaves the bulk geometry unchanged. Finally, when only one type of superfield is present, the geometry is Kähler.

3.2 From $N = (2,2)$ to $N = 2$

We introduce a boundary in $N = (2,2)$ superspace which breaks half of the supersymmetries, reducing $N = (2,2)$ to $N = 2$. We have either B-type boundary conditions where the boundary is given by $\theta' \equiv (\theta^+ - \theta^-)/2 = 0$ and $\hat{\theta}' \equiv (\hat{\theta}^+ - \hat{\theta}^-)/2 = 0$ or A-type boundary conditions where the boundary is given by $\theta' \equiv (\theta^+ - \theta^-)/2 = 0$ and $\hat{\theta}' \equiv (\hat{\theta}^+ + \hat{\theta}^-)/2 = 0$. Throughout this paper we will always use B-type boundary conditions as switching to A-type boundary conditions merely amounts to interchanging chiral fields for twisted chiral fields and vice-versa [19].

We define the derivatives,

$$\begin{aligned} D &\equiv D_+ + D_-, \qquad & \hat{D} &\equiv \hat{D}_+ + \hat{D}_-, \\ D' &\equiv D_+ - D_-, \qquad & \hat{D}' &\equiv \hat{D}_+ - \hat{D}_-, \end{aligned} \tag{3.12}$$

where unaccented derivatives refer to translations in the invariant directions. They satisfy,

$$\begin{aligned} D^2 = \hat{D}^2 = D'^2 = \hat{D}'^2 &= -\frac{i}{2}\partial_\tau, \\ \{D, D'\} = \{\hat{D}, \hat{D}'\} &= -i\partial_\sigma, \end{aligned} \tag{3.13}$$

with all other anti-commutators zero.

Let us now turn to the superfields. In the bulk we had chiral and twisted chiral superfields. From eqs. (3.6) and (3.12) we get for the chiral fields,

$$\begin{aligned} \hat{D}X^\alpha &= +iDX^\alpha, \qquad & \hat{D}X^{\bar{\alpha}} &= -iDX^{\bar{\alpha}}, \\ \hat{D}'X^\alpha &= +iD'X^\alpha, \qquad & \hat{D}'X^{\bar{\alpha}} &= -iD'X^{\bar{\alpha}}, \end{aligned} \tag{3.14}$$

where $\alpha,\ \bar{\alpha} \in \{1, \ldots, m\}$. This can also be written as,[8]

$$\bar{\mathbb{D}}X^\alpha = \bar{\mathbb{D}}'X^\alpha = \mathbb{D}X^{\bar{\alpha}} = \mathbb{D}'X^{\bar{\alpha}} = 0. \tag{3.15}$$

[7] Indices from the beginning of the Greek alphabet, α, β, γ, ... denote chiral fields while indices from the middle of the alphabet, μ, ν, ρ, ... denote twisted chiral fields.

[8] For our conventions we refer once more to the appendix.

JHEP10(2008)108

Passing from $N = (2,2)$ — parameterized by the Grassmann coordinates θ, $\hat{\theta}$, θ' and $\hat{\theta}'$ — to $N = 2$ superspace – parameterized by θ and $\hat{\theta}$ — we get X^α, $X^{\bar{\alpha}}$, $D'X^\alpha$ and $D'X^{\bar{\alpha}}$ as $N = 2$ superfields which satisfy the constraints,

$$\begin{aligned} \hat{D}X^\alpha &= +i\,DX^\alpha, & \hat{D}X^{\bar{\alpha}} &= -i\,DX^{\bar{\alpha}}, \\ \hat{D}\,D'X^\alpha &= +i\,D\,D'X^\alpha - \partial_\sigma X^\alpha, & \hat{D}\,D'X^{\bar{\alpha}} &= -i\,D\,D'X^{\bar{\alpha}} + \partial_\sigma X^{\bar{\alpha}}. \end{aligned} \tag{3.16}$$

For twisted chiral superfields we get instead, when combining eqs. (3.7) and (3.12),

$$\begin{aligned} \hat{D}X^\mu &= +iD'X^\mu, & \hat{D}X^{\bar{\mu}} &= -iD'X^{\bar{\mu}}, \\ \hat{D}'X^\mu &= +iDX^\mu, & \hat{D}'X^{\bar{\mu}} &= -iDX^{\bar{\mu}}, \end{aligned} \tag{3.17}$$

with $\mu,\ \bar{\mu} \in \{1, \ldots, n\}$. For further convenience we can also write this as,

$$\begin{aligned} \mathbb{D}'X^\mu &= \mathbb{D}X^\mu, & \bar{\mathbb{D}}'X^\mu &= -\bar{\mathbb{D}}X^\mu, \\ \mathbb{D}'X^{\bar{\mu}} &= -\mathbb{D}X^{\bar{\mu}}, & \bar{\mathbb{D}}'X^{\bar{\mu}} &= \bar{\mathbb{D}}X^{\bar{\mu}}. \end{aligned} \tag{3.18}$$

Passing again from $N = (2,2)$ to $N = 2$ superspace, we now get X^μ, $X^{\bar{\mu}}$, $D'X^\mu$ and $D'X^{\bar{\mu}}$ as $N = 2$ superfields satisfying the constraints,

$$\begin{aligned} \hat{D}X^\mu &= +i\,D'X^\mu, & \hat{D}X^{\bar{\mu}} &= -i\,D'X^{\bar{\mu}}, \\ \hat{D}\,D'X^\mu &= -\frac{1}{2}\dot{X}^\mu, & \hat{D}\,D'X^{\bar{\mu}} &= +\frac{1}{2}\dot{X}^{\bar{\mu}}. \end{aligned} \tag{3.19}$$

Note that in $N = 2$ boundary superspace, the twisted chiral superfields X^μ and $X^{\bar{\mu}}$ are unconstrained superfields — the fermionic fields $D'X$ are nothing else but the image of these fields under the second supersymmetry — while the chiral fields can be viewed as a 1-d analogue of chiral fields.

Once more one immediately verifies that the difference between the fermionic measure $D_+D_-\hat{D}_+\hat{D}_-$ and $D\hat{D}D'\hat{D}'$ is just a boundary term. So the most general $N = 2$ invariant action which reduces to the usual action away from the boundary that we can write down is,

$$\begin{aligned} \mathcal{S} &= -\int d^2\sigma\, d\theta d\hat{\theta}\, D'\hat{D}'\, V(X,\bar{X}) + i\int d\tau\, d\theta d\hat{\theta}\, W(X,\bar{X}) \\ &= -\int d^2\sigma\, d^2\theta\, D'\hat{D}'\, V(X,\bar{X}) + i\int d\tau\, d^2\theta\, W(X,\bar{X}), \end{aligned} \tag{3.20}$$

with $V(X,\bar{X})$ and $W(X,\bar{X})$ real functions of the chiral and the twisted chiral superfields. While the generalized Kähler potential V is arbitrary, this is not so for the boundary potential W. Whenever W is a function of the twisted chiral fields as well, its form will be (partially) determined by the boundary conditions as we will see later on.

4. Non-linear σ-models

4.1 The action

We start with a set of chiral superfields X^α, $\alpha \in \{1, \ldots m\}$, and a set of twisted chiral superfields X^μ, $\mu \in \{1, \ldots, n\}$. The action is given by eq. (3.20). Working out the $\hat{D}'$ and

D' derivatives we obtain the action in $N = 2$ superspace,[9]

$$
\begin{aligned}
\mathcal{S} = \int d^2\sigma\, d^2\theta \, \big\{ & + 2i\, V_{\alpha\bar\beta} D'X^\alpha D'X^{\bar\beta} - 2i\, V_{\alpha\bar\beta} DX^\alpha DX^{\bar\beta} + 2i\, V_{\mu\bar\nu} DX^\mu D'X^{\bar\nu} \\
& + 2i\, V_{\mu\bar\nu} D'X^\mu DX^{\bar\nu} - V_\mu \partial_\sigma X^\mu + V_{\bar\mu}\partial_\sigma X^{\bar\mu} - V_\alpha \partial_\sigma X^\alpha + V_{\bar\alpha}\partial_\sigma X^{\bar\alpha} \\
& + 2i\, V_{\alpha\bar\beta} D'X^\alpha DX^{\bar\beta} + 2i\, V_{\alpha\bar\beta} DX^\alpha D'X^{\bar\beta} + 2i\, V_{\alpha\bar\nu} D'X^\alpha D'X^{\bar\nu} \\
& + 2i\, V_{\mu\bar\beta} D'X^\mu D'X^{\bar\beta} + 2i\, V_{\mu\bar\beta} DX^\mu D'X^{\bar\beta} + 2i\, V_{\alpha\bar\nu} D'X^\alpha DX^{\bar\nu} \big\} \\
+ i \int d\tau\, d^2\theta\, W(X, \bar X). &
\end{aligned}
\tag{4.1}
$$

When reducing the action to $N = 1$ superspace, one recovers the action in eq. (2.7) with metric and torsion given by eq. (3.11). However the resulting action has a boundary term as well,

$$
\begin{aligned}
\mathcal{S}_{\text{boundary}} = -i \int d\tau\, d\theta \Big(& \big(V_\alpha - iW_\alpha\big) DX^\alpha + \big(V_{\bar\alpha} + iW_{\bar\alpha}\big) DX^{\bar\alpha} + \\
& + \big(V_\mu - iW_\mu\big) D'X^\mu + \big(V_{\bar\mu} + iW_{\bar\mu}\big) D'X^{\bar\mu} \Big).
\end{aligned}
\tag{4.2}
$$

Note that this boundary term — because of the presence of $D'X$ terms – does not have the standard form (compare to eq. (2.9)). A judicious choice of boundary conditions will allow us to reduce it to the form in eq. (2.9).

The action is still invariant under the generalized Kähler transformations, eqs. (3.9) and (3.10), provided the boundary potential W transforms as well,

$$
W \to W - i\big(F - \bar F\big) + i\big(G - \bar G\big). \tag{4.3}
$$

Invariance under generalized Kähler transformations is essential for the global consistency of the models. Let us illustrate this with a simple example — more and less trivial examples will follow later in the paper – of a D1-brane on a two-torus T^2. The torus is characterized by its modulus τ which takes its value in the upper half-plane $\mathbb{H}$. We parametrize the torus by a twisted chiral field $w = (x + \tau y)/\sqrt{2}$ with $x, y \in \mathbb{R}$, such that the metric is simply $g_{w\bar w} = 1$. The periodicity condition is,

$$
w \simeq w + \frac{1}{\sqrt{2}}\big(n_1 + n_2\,\tau\big), \tag{4.4}
$$

with $n_1, n_2 \in \mathbb{Z}$. We impose the Dirichlet boundary condition,

$$
\big(1 + m\,\bar\tau\big)\, w = \big(1 + m\,\tau\big)\, \bar w, \tag{4.5}
$$

with $m \in \mathbb{Z}$. Because of eq. (3.19) this implies a Neumann boundary condition as well,

$$
\big(1 + m\,\bar\tau\big)\, D'w + \big(1 + m\,\tau\big)\, D'\bar w = 0, \tag{4.6}
$$

and we end up with a D1-brane winding once in the x direction and m times in the y direction. The Kähler potential is $V = -w\bar w$ and with the boundary condition eq. (4.5)

[9]When comparing this to the Kähler case discussed in [19], note that when no twisted chiral fields are present, $\int d^2\sigma d^2\theta V_\alpha \partial_\sigma X^\alpha = -2i \int d^2\sigma d^2\theta V_{\bar\alpha\beta} DX^{\bar\alpha} D'X^\beta$ holds.

one finds[10] that the boundary potential vanishes, $W = 0$. Because of the presence of a D1-brane, the invariance eq. (4.4) is partially broken and we get from eq. (4.5) that,

$$n_2 = m\, n_1, \tag{4.7}$$

should hold. Under eqs. (4.4) and (4.7), the Kähler potential transforms as,

$$V \to V - \frac{n_1}{\sqrt{2}}(1 + m\,\bar{\tau})w - \frac{n_1}{\sqrt{2}}(1 + m\,\tau)\bar{w}. \tag{4.8}$$

Making a Kähler transformation restores the invariance but generates – because of eq. (4.3) — a boundary potential,

$$W = 0 \to W = -\frac{i\, n_1}{\sqrt{2}}\left(\big(1 + m\,\bar{\tau}\big)\, w - \big(1 + m\,\tau\big)\, \bar{w}\right), \tag{4.9}$$

which vanishes because of the boundary condition eq. (4.5). So the description is indeed globally consistent.

Finally note that the action eq. (3.20) is also invariant under,

$$W \to W + H + \bar{H}, \tag{4.10}$$

where,

$$\partial_{\bar{\alpha}} H = \partial_{\mu} H = \partial_{\bar{\mu}} H = 0, \qquad \partial_{\alpha} \bar{H} = \partial_{\mu} \bar{H} = \partial_{\bar{\mu}} \bar{H} = 0. \tag{4.11}$$

We will often tacitly use the fact that the boundary potential is only defined modulo an additive contribution of a holomorphic (and its complex conjugate) function of the boundary chiral fields.

When varying the action, one needs to take into account that the superfields are constrained. Besides X^{μ} and $X^{\bar{\mu}}$ we introduce unconstrained superfields Λ^{α}, $\Lambda^{\bar{\alpha}}$, M^{α} and $M^{\bar{\alpha}}$ and solve the constraints by,

$$\begin{aligned}
X^{\alpha} &= \bar{\mathbb{D}}\Lambda^{\alpha}, & X^{\bar{\alpha}} &= \mathbb{D}\Lambda^{\bar{\alpha}}, \\
D' X^{\alpha} &= \bar{\mathbb{D}} M^{\alpha} - \partial_{\sigma}\Lambda^{\alpha}, & D' X^{\bar{\alpha}} &= \mathbb{D} M^{\bar{\alpha}} + \partial_{\sigma}\Lambda^{\bar{\alpha}}, \\
D' X^{\mu} &= -i\,\hat{D} X^{\mu}, & D' X^{\bar{\mu}} &= +i\hat{D} X^{\bar{\mu}}.
\end{aligned} \tag{4.12}$$

Varying the unconstrained fields in the action eq. (4.1) yields the usual equations of motion with metric and Kalb-Ramond two-form given by eq. (3.11) and a boundary term given by,

$$\begin{aligned}
\delta\mathcal{S}\Big|_{\text{boundary}} = \int d\tau\, d^2\theta\, \Big\{ & \delta\Lambda^{\alpha}\big(\bar{\mathbb{D}}' V_{\alpha} + i\,\bar{\mathbb{D}} W_{\alpha}\big) - \delta\Lambda^{\bar{\alpha}}\big(\mathbb{D}' V_{\bar{\alpha}} - i\,\mathbb{D} W_{\bar{\alpha}}\big) + \\
& - \delta X^{\mu}\big(V_{\mu} - iW_{\mu}\big) + \delta X^{\bar{\mu}}\big(V_{\bar{\mu}} + iW_{\bar{\mu}}\big)\Big\}.
\end{aligned} \tag{4.13}$$

This should vanish by imposing appropriate boundary conditions on the fields.

[10] One can use the results in [19] or require that eq. (4.13) vanishes.

4.2 Boundary conditions

4.2.1 General considerations

We will impose Dirichlet boundary conditions using an almost product structure as was introduced in section 2. We start with the unconstrained superfields Λ. The most general Dirichlet boundary conditions which are consistent with the dimensions of the fields involved are given by,

$$\delta\Lambda^\alpha = R(X)^\alpha{}_\beta \delta\Lambda^\beta + R(X)^\alpha{}_{\bar\beta}\delta\Lambda^{\bar\beta}, \tag{4.14}$$

which already implies that,

$$R^\alpha{}_\mu = R^\alpha{}_{\bar\mu} = 0. \tag{4.15}$$

As $\bar{\mathbb{D}}\delta\Lambda^{\bar\beta}$ should not appear in the boundary condition for X^α we necessarily have that,

$$R^\alpha{}_{\bar\beta} = 0. \tag{4.16}$$

Eq. (4.14) implies,

$$\delta X^\alpha = R^\alpha{}_\beta\, \delta X^\beta, \tag{4.17}$$

if,

$$\begin{aligned} R^\alpha{}_{\delta,\bar\epsilon}\mathcal{P}^\delta_{+\beta}\mathcal{P}^{\bar\epsilon}_{+\bar\gamma} + R^\alpha{}_{\delta,\mu}\mathcal{P}^\delta_{+\beta}\mathcal{P}^\mu_{+\bar\gamma} + R^\alpha{}_{\delta,\bar\mu}\mathcal{P}^\delta_{+\beta}\mathcal{P}^{\bar\mu}_{+\bar\gamma} &= 0,\\ R^\alpha{}_{\delta,\nu}\mathcal{P}^\delta_{+\beta}\mathcal{P}^\nu_{+\mu} + R^\alpha{}_{\delta,\bar\nu}\mathcal{P}^\delta_{+\beta}\mathcal{P}^{\bar\nu}_{+\mu} &= 0,\\ R^\alpha{}_{\delta,\nu}\mathcal{P}^\delta_{+\beta}\mathcal{P}^\nu_{+\bar\mu} + R^\alpha{}_{\delta,\bar\nu}\mathcal{P}^\delta_{+\beta}\mathcal{P}^{\bar\nu}_{+\bar\mu} &= 0, \end{aligned} \tag{4.18}$$

holds. Furthermore, eq. (4.17) implies $DX^\alpha = R^\alpha{}_\beta\, DX^\beta$, $\hat D X^\alpha = R^\alpha{}_\beta\, \hat D X^\beta$ and $\dot X^\alpha = R^\alpha{}_\beta\, \dot X^\beta$ as well. Consistency of this with $D^2 = \hat D^2 = -(i/2)\partial_\tau$ results in the integrability condition,

$$R^\alpha{}_{\delta,\epsilon}\, \mathcal{P}_+{}^\delta{}_{[\beta}\, \mathcal{P}_+{}^\epsilon{}_{\gamma]} = 0. \tag{4.19}$$

Eqs. (4.18) and (4.19) together form the integrability conditions (2.17) for $a = \alpha$ and thus guarantee the existence of a holomorphic coordinate transformation which brings us to coordinates $X^{\hat\alpha}$, $\hat\alpha \in \{k+1, \dots m\}$ where $2k$ is the rank of $\mathcal{P}_+$ (in the chiral directions), such that part of the Dirichlet boundary conditions are given by,

$$X^{\hat\alpha} = \text{ constant}, \tag{4.20}$$

with $X^{\hat\alpha}$ chiral. We will denote the remainder of the chiral coordinates by $X^{\tilde\alpha}$, $\tilde\alpha \in \{1, \dots, k\}$. In these coordinates we have that,

$$R^{\hat\alpha}{}_{\tilde\beta} = 0, \qquad R^{\tilde\alpha}{}_{\tilde\beta} = \delta^{\tilde\alpha}_{\tilde\beta}, \tag{4.21}$$

and,

$$R^{\hat\alpha}{}_{\hat\gamma} R^{\hat\gamma}{}_{\hat\beta} = \delta^{\hat\alpha}_{\hat\beta}, \qquad R^{\tilde\alpha}{}_{\hat\gamma} R^{\hat\gamma}{}_{\hat\beta} = -R^{\tilde\alpha}{}_{\hat\beta}. \tag{4.22}$$

For the time being however, we only require the chiral fields to obey (4.17), without going to these adapted coordinates.

We now turn to the Dirichlet boundary conditions for the twisted chiral superfields. The most general expression we can write down is,

$$\delta X^{\mu} = R^{\mu}{}_{\nu}\,\delta X^{\nu} + R^{\mu}{}_{\bar{\nu}}\,\delta X^{\bar{\nu}} + R^{\mu}{}_{\beta}\,\delta X^{\beta} + R^{\mu}{}_{\bar{\beta}}\,\delta X^{\bar{\beta}}. \tag{4.23}$$

Using eqs. (3.19) and (3.16), we get from this,[11]

$$\left(\mathcal{P}_{+}D'X\right)^{\mu} = R^{\mu}{}_{\nu}\,D'X^{\nu} + \frac{1}{2}R^{\mu}{}_{\beta}\left(D'X^{\beta} + DX^{\beta}\right) + \frac{1}{2}R^{\mu}{}_{\bar{\beta}}\left(D'X^{\bar{\beta}} - DX^{\bar{\beta}}\right). \tag{4.24}$$

This is consistent with $\mathcal{P}_{+}^{2} = \mathcal{P}_{+}$ if,

$$\begin{aligned} &R^{\mu}{}_{\rho}R^{\rho}{}_{\nu} = R^{\mu}{}_{\nu}, \\ &R^{\mu}{}_{\rho}R^{\rho}{}_{\bar{\nu}} = R^{\mu}{}_{\bar{\rho}}R^{\bar{\rho}}{}_{\bar{\nu}} = 0, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} R^{\mu}{}_{\beta} &= R^{\mu}{}_{\nu}R^{\nu}{}_{\beta} - R^{\mu}{}_{\bar{\nu}}R^{\bar{\nu}}{}_{\beta}, \\ R^{\mu}{}_{\bar{\beta}} &= R^{\mu}{}_{\nu}R^{\nu}{}_{\bar{\beta}} - R^{\mu}{}_{\bar{\nu}}R^{\bar{\nu}}{}_{\bar{\beta}}. \end{aligned} \tag{4.26}$$

Using the defining property of an almost product structure – $R^{a}{}_{c}R^{c}{}_{b} = R^{a}{}_{b}$ — and eqs. (4.15) and (4.25), one finds that both $\pi^{\mu}_{+\nu} \equiv R^{\mu}{}_{\nu}$ and $\pi^{\mu}_{-\nu} \equiv \delta^{\mu}_{\nu} - R^{\mu}{}_{\nu} = R^{\mu}{}_{\bar{\rho}}R^{\bar{\rho}}{}_{\nu}$ are projection operators mapping $T^{(1,0)}_{\mathcal{M}}$ to $T^{(1,0)}_{\mathcal{M}}$. In terms of these projection operators eqs. (4.26) can be rewritten more suggestively as,

$$\begin{aligned} R^{\mu}{}_{\beta} &= \left(\pi^{\mu}_{-\nu} - \pi^{\mu}_{+\nu}\right) R^{\nu}{}_{\alpha}R^{\alpha}{}_{\beta}, \\ R^{\mu}{}_{\bar{\beta}} &= \left(\pi^{\mu}_{-\nu} - \pi^{\mu}_{+\nu}\right) R^{\nu}{}_{\bar{\alpha}}R^{\bar{\alpha}}{}_{\bar{\beta}}. \end{aligned} \tag{4.27}$$

In the π_{-} directions these relations are trivially satisfied in chiral directions along the brane, as follows from (4.17) or (4.21), and hence have no consequences for the Dirichlet conditions (4.23). In the π_{+} directions, however, they imply,

$$\pi^{\mu}_{+\nu}R^{\nu}{}_{\beta}\delta X^{\beta} = \pi^{\mu}_{+\nu}R^{\nu}{}_{\bar{\beta}}\delta X^{\bar{\beta}} = 0, \tag{4.28}$$

or

$$\pi^{\mu}_{+\nu}R^{\nu}{}_{\tilde{\beta}} = \pi^{\mu}_{+\nu}R^{\nu}{}_{\bar{\tilde{\beta}}} = 0. \tag{4.29}$$

This implies that there are no Dirichlet conditions in the π_{+} directions, as can be seen by acting with π_{+} on both sides of eq. (4.23). This is made manifest by writing the Dirichlet boundary conditions eq. (4.23) as

$$\pi^{\mu}_{-\nu}\delta X^{\nu} = R^{\mu}{}_{\bar{\nu}}\,\delta X^{\bar{\nu}} + R^{\mu}{}_{\tilde{\beta}}\,\delta X^{\tilde{\beta}} + R^{\mu}{}_{\bar{\tilde{\beta}}}\,\delta X^{\bar{\tilde{\beta}}}. \tag{4.30}$$

[11] Note that the contributions from $DX^{\hat{\alpha}}$ (and its complex conjugate) to this expression actually vanish because of (4.17), but the ones from $D'X^{\hat{\alpha}}$ (and its complex conjugate)do not.

The corresponding Neumann boundary conditions are then,

$$\pi^{\mu}_{-\nu} D'X^{\nu} = -R^{\mu}{}_{\bar\nu}\, D'X^{\bar\nu} + R^{\mu}{}_{\tilde\beta}\, DX^{\tilde\beta} - R^{\mu}{}_{\bar{\tilde\beta}}\, DX^{\bar{\tilde\beta}}. \tag{4.31}$$

We need to impose separate Neumann boundary conditions on $\pi^{\mu}_{+\nu} D'X^{\nu}$. In the π_+ directions (4.24) can be written as,

$$\pi^{\mu}_{+\nu}\big(\mathcal{P}_+ D'X\big)^{\nu} = R^{\mu}{}_{\nu}\left(D'X^{\nu} + \frac{1}{2}R^{\nu}{}_{\hat\beta}\, D'X^{\hat\beta} + \frac{1}{2}R^{\nu}{}_{\bar{\hat\beta}}\, D'X^{\bar{\hat\beta}}\right), \tag{4.32}$$

which, comparing to e.g. eq. (2.19), shows that there is a non-degenerate $\mathrm{U}(1)$ field strength in the π_+ directions. A similar expression in the π_- directions

$$\pi^{\mu}_{-\nu}\big(\mathcal{P}_+ D'X\big)^{\nu} = \frac{1}{2}\pi^{\mu}_{-\nu}\left(R^{\nu}{}_{\beta}\left(D'X^{\beta} + DX^{\beta}\right) + R^{\nu}{}_{\bar\beta}\left(D'X^{\bar\beta} - DX^{\bar\beta}\right)\right), \tag{4.33}$$

indicates that we can expect a field strength in these directions as well, as long as $R^{\mu}{}_{\beta}$ and $R^{\mu}{}_{\bar\beta}$ are non-vanishing.

4.2.2 Detailed analysis

For simplicity, let us first examine the extremal cases $\pi_- = 1$ and $\pi_+ = 1$. When $\pi_- = 1$, we get an equal amount of Dirichlet and Neumann conditions on the twisted chiral fields. To make (4.13) vanish, we start by setting,

$$(V - iW)_{\mu} R^{\mu}{}_{\bar\nu} = (V + iW)_{\bar\nu}. \tag{4.34}$$

Using this, one rewrites the Dirichlet conditions eq. (4.30) as,

$$(V - iW)_{\mu}\left(\delta X^{\mu} - R^{\mu}{}_{\beta}\delta X^{\beta}\right) = (V + iW)_{\bar\mu}\left(\delta X^{\bar\mu} - R^{\bar\mu}{}_{\bar\beta}\delta X^{\bar\beta}\right), \tag{4.35}$$

which also implies $R^{\mu}{}_{\beta} = -R^{\mu}{}_{\bar\nu}R^{\bar\nu}{}_{\beta}$ or,

$$(V - iW)_{\mu} R^{\mu}{}_{\beta} = -(V + iW)_{\bar\mu} R^{\bar\mu}{}_{\beta}. \tag{4.36}$$

Compatibility of eq. (4.35) with $D^2 \propto \partial_\tau$ results in the extra condition

$$R_{\mu\nu} = R_{\nu\mu}, \tag{4.37}$$

along with the conditions which insure integrability of $\mathcal{P}_+$, eq. (2.17), already known from the $N = 1$ superspace analysis. The Dirichlet conditions (4.35) again automatically imply the Neumann conditions

$$(V - iW)_{\mu}\left(D'X^{\mu} - R^{\mu}{}_{\beta}DX^{\beta}\right) + (V + iW)_{\bar\mu}\left(D'X^{\bar\mu} - R^{\bar\mu}{}_{\bar\beta}DX^{\bar\beta}\right) = 0. \tag{4.38}$$

Using these equations to simplify the $N = 1$ boundary term (4.2) and comparing the result with (2.9) yields a $\mathrm{U}(1)$ connection with components

$$\begin{aligned}
A_{\tilde\alpha} &= -\frac{1}{2}\left[(V - iW)_{\tilde\alpha} + (V - iW)_{\mu} R^{\mu}{}_{\tilde\alpha}\right],\\
A_{\bar{\tilde\alpha}} &= -\frac{1}{2}\left[(V + iW)_{\bar{\tilde\alpha}} + (V + iW)_{\bar\mu} R^{\bar\mu}{}_{\bar{\tilde\alpha}}\right],\\
A_{\mu} &= A_{\bar\mu} = 0,
\end{aligned} \tag{4.39}$$

up to gauge transformations. Using (4.35) in (4.13) yields,

$$\delta\mathcal{S}\Big|_{\text{boundary}} = \int d\tau\, d^2\theta \Big\{\delta\Lambda^{\alpha}\Big[\bar{\mathbb{D}}'V_{\alpha} - \bar{\mathbb{D}}\big((V-iW)_{\mu}R^{\mu}{}_{\alpha} - i\,W_{\alpha}\big)\Big] \\ -\delta\Lambda^{\bar{\alpha}}\Big[\mathbb{D}'V_{\bar{\alpha}} - \mathbb{D}\big((V+iW)_{\bar{\mu}}R^{\bar{\mu}}{}_{\bar{\alpha}} + i\,W_{\bar{\alpha}}\big)\Big]\Big\}. \quad (4.40)$$

This leads to the Neumann conditions

$$\bar{\mathbb{D}}'V_{\tilde{\alpha}} = \bar{\mathbb{D}}\big((V-iW)_{\mu}R^{\mu}{}_{\tilde{\alpha}} - i\,W_{\tilde{\alpha}}\big), \\ \mathbb{D}'V_{\bar{\tilde{\alpha}}} = \mathbb{D}\big((V+iW)_{\bar{\mu}}R^{\bar{\mu}}{}_{\bar{\tilde{\alpha}}} + i\,W_{\bar{\tilde{\alpha}}}\big). \quad (4.41)$$

Eqs. (4.41) together with (4.20), (4.35) and (4.38), describe a $(2k+n)$-dimensional brane, where k is the number of chiral fields along the brane and n is the number of twisted chiral fields. In principle, it should be possible to demonstrate that (4.38) and (4.41) are precisely of the general form (2.19). This however requires the introduction of worldvolume coordinates (which solve eqs. (4.35)). We will not show this in full generality here, but will illustrate this point for some examples in section 5.

Let us now turn to the case $\pi_+ = 1$. As mentioned before, in this case all boundary conditions on the twisted chiral fields are necessarily Neumann. Because of the presence of a U(1) field strength F — as implied by (4.32) — we expect a condition of the form

$$D'X^k = F^k{}_l DX^l + (F^k{}_a + b^k{}_a)DX^a. \quad (4.42)$$

Here, Latin indices from the beginning of the alphabet indicate (both holomorphic and anti-holomorphic) chiral directions and Latin indices from the middle of the alphabet indicate twisted chiral directions. Note that we have taken into account the non-trivial b-field background (3.11). Writing the $N=2$ relations as $\hat{D}X^a = J^a{}_b DX^b$ and $\hat{D}X^m = J^m{}_n D'X^n$, we find that eq. (4.42) implies,[12]

$$\hat{D}X^k = K^k{}_l DX^l + L^k{}_a DX^a, \quad (4.43)$$

where $K^k{}_l = J^k{}_m F^m{}_l$ and $L^k{}_a = J^k{}_m(F^m{}_a + b^m{}_a)$. This means that on the boundary the twisted chiral fields become constrained superfields. Consistency of these constraints with $\hat{D}^2 = D^2 \propto \partial_\tau$, implies that K is a(n integrable) complex structure on the space spanned by the twisted chiral fields, while L should satisfy one set of algebraic relations,

$$K^k{}_l L^l{}_a = -L^k{}_b J^b{}_a, \quad (4.44)$$

and two sets of relations involving derivatives,

$$0 = K^k{}_m K^m{}_{l,a} - K^k{}_m L^m{}_{a,l} + K^m{}_l L^k{}_{a,m} - L^m{}_a K^k{}_{l,m} - J^b{}_a K^k{}_{l,b}, \\ 0 = K^k{}_l L^l{}_{[a,b]} + L^l{}_{[a} L^k{}_{b],l} + J^c{}_{[a} L^k{}_{b],c}. \quad (4.45)$$

[12]Notice that components $L^k{}_{\hat{a}}$ will always be zero, since there is no magnetic field in these directions. This is however implicit in all subsequent formulae, because in the end $DX^{\hat{a}}$ will be zero as well by virtue of the Dirichlet conditions in the chiral directions.

JHEP10(2008)108

This can be interpreted as follows. We can combine the constraints on the chiral fields and eq. (4.43) in a straightforward way to the following constraint on the boundary,

$$\hat{D}\begin{pmatrix} X^a \\ X^k \end{pmatrix} = \begin{pmatrix} J^a{}_b & 0 \\ L^k{}_b & K^k{}_l \end{pmatrix} D \begin{pmatrix} X^b \\ X^l \end{pmatrix}. \tag{4.46}$$

The above conditions on K and L are then nothing but the statement that the matrix

$$\mathcal{K} = \begin{pmatrix} J & 0 \\ L & K \end{pmatrix} \tag{4.47}$$

represents a complex structure, $\mathcal{K}^2 = -1$ and $\mathcal{N}^M{}_{NK}(\mathcal{K},\mathcal{K}) = 0$. Here we introduced indices M, $N, \dots$ which run over both chiral Neumann directions and twisted chiral directions. In terms of these, we can thus write

$$\hat{D}X^M = \mathcal{K}^M{}_N DX^N. \tag{4.48}$$

Because of (4.48), the X^M are not all independent. In order to deal with this when considering the boundary term in variation of the action, these constraints are again solved by introducing fermionic superfields $\tilde{\Lambda}^M$ such that

$$\delta X^M = \frac{\partial X^M}{\partial \tilde{X}^N}\left(\hat{D}\delta\tilde{\Lambda}^N - \tilde{\mathcal{K}}^N{}_P D\delta\tilde{\Lambda}^P\right), \tag{4.49}$$

where $\tilde{X}$ are coordinates with respect to which $\tilde{\mathcal{K}}$ is constant.[13] Note that the chiral component of (4.49) is nothing but

$$\delta X^a = \hat{D}\delta\Lambda^a - J^a{}_b D\delta\Lambda^b, \tag{4.50}$$

where (because both $J^a{}_b$ and $\tilde{J}^a{}_b$ are constant)

$$\delta\Lambda^a = \frac{\partial X^a}{\partial \tilde{X}^M}\delta\tilde{\Lambda}^M. \tag{4.51}$$

This shows that the $\delta\Lambda^a$ are exactly the unconstrained superfields needed to obtain (4.13), so that this construction is consistent with previously obtained expressions. We now write the second line of (4.13) as,

$$i\int d\tau\, d^2\theta\, M_k\delta X^k = i\int d\tau\, d^2\theta\,(M_N\delta X^N - M_a\delta X^a), \tag{4.52}$$

where we introduced $M_N = W_N + V_M J^M{}_N$. A calculation formally identical to the one leading to eq. (4.42) of [19] then yields

$$i\int d\tau\, d^2\theta\, M_N\delta X^N = i\int d\tau\, d^2\theta\, \delta\Lambda^M DX^N\left(M_{M,P}\mathcal{K}^P{}_N - M_{P,N}\mathcal{K}^P{}_M + 2M_P\mathcal{K}^P{}_{[N,M]}\right),$$

[13]Note that the coordinates $\tilde{X}$ need not separate nicely into a set of chiral and a set of twisted chiral superfields. In the end, the boundary term in the variation will again be expressed in terms of the coordinates $X^{\tilde{a}}$ and X^k.

where we used (4.49) and we introduced the notation

$$\delta\Lambda^M = \frac{\partial X^M}{\partial \tilde{X}^N}\delta\tilde{\Lambda}^N. \tag{4.53}$$

The second term of (4.52) is easier to work out in a similar way using (4.50). Putting all pieces together, one arrives at the following expression for the boundary term in the variation of the action (4.13):

$$\delta\mathcal{S}\Big|_{\text{boundary}} = \mathcal{I}_A + \mathcal{I}_B, \tag{4.54}$$

where

$$\begin{aligned}\mathcal{I}_A = &-i\int d\tau\, d^2\theta\, \delta\Lambda^a\left(\hat{D}'V_b J^b{}_a + D'V_a - \hat{D}W_a + DW_b J^b{}_a + DM_k L^k{}_a\right)\\ &-i\int d\tau\, d^2\theta\, \delta\Lambda^a\left(2M_k L^k{}_{[a,b]}DX^b + M_l L^l{}_{a,k}DX^k - M_l K^l{}_{k,a}DX^k\right),\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}\mathcal{I}_B = &\, i\int d\tau\, d^2\theta\, \delta\Lambda^k\left(\hat{D}M_k - DM_l K^l{}_k\right) +\\ &\, i\int d\tau\, d^2\theta\, \delta\Lambda^k\left(2M_m K^m{}_{[l,k]}DX^l + M_l L^l{}_{a,k}DX^a - M_l K^l{}_{k,a}DX^a\right).\end{aligned} \tag{4.56}$$

The first term $\mathcal{I}_A$ disappears when imposing Dirichlet conditions,

$$\delta\Lambda^{\hat{a}} = 0, \tag{4.57}$$

and Neumann conditions,

$$\begin{aligned}0 = &\, \hat{D}'V_{\tilde{b}}J^{\tilde{b}}{}_{\tilde{a}} + D'V_{\tilde{a}} - \hat{D}W_{\tilde{a}} + DW_{\tilde{b}}J^{\tilde{b}}{}_{\tilde{a}} + DM_k L^k{}_{\tilde{a}}\\ &+2M_k L^k{}_{[\tilde{a},\tilde{b}]}DX^{\tilde{b}} + M_l(L^l{}_{\tilde{a},k} - K^l{}_{k,\tilde{a}})DX^k.\end{aligned} \tag{4.58}$$

The second term $\mathcal{I}_B$ vanishes if we impose

$$0 = M_{k,m}K^m{}_l - M_{m,l}K^m{}_k + 2M_m K^m{}_{[l,k]}, \tag{4.59}$$

$$0 = M_{k,\tilde{b}}J^{\tilde{b}}{}_{\tilde{a}} - M_{l,\tilde{a}}K^l{}_k + M_{k,l}L^l{}_{\tilde{a}} + M_l L^l{}_{\tilde{a},k} - M_l K^l{}_{k,\tilde{a}}. \tag{4.60}$$

Using the fact that

$$F_{kl} = -g_{km}(JK)^m{}_l, \tag{4.61}$$

and the boundary condition (4.59), we find that $F_{kl} = \partial_k A_l - \partial_l A_k$, where

$$A_k = \frac{1}{2}M_l K^l{}_k + \partial_k f, \tag{4.62}$$

with f an arbitrary real function. This shows that the U(1) gauge fields in the twisted chiral directions are unaltered with respect to the case where only twisted chiral fields are present [19]. On the other hand, we have that

$$F_{k\tilde{a}} = -g_{kl}(JL)^l{}_{\tilde{a}} - b_{k\tilde{a}}, \tag{4.63}$$

where, again, the b-field is given by (3.11). This together with (4.60) implies that $F_{k\tilde{a}} = \partial_k A_{\tilde{a}} - \partial_{\tilde{a}} A_k$, where A_k is again given by (4.62) and $A_{\tilde{a}}$ can be written as,

$$A_{\tilde{a}} = \frac{1}{2} M_k L^k{}_{\tilde{a}} + \frac{1}{2} M_{\tilde{b}} J^{\tilde{b}}{}_{\tilde{a}} + \partial_{\tilde{a}} f. \tag{4.64}$$

Eqs. (4.62) and (4.64) can be summarized by using the complex structure $\mathcal{K}$,

$$\begin{pmatrix} A_{\tilde{a}} & A_k \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} M_{\tilde{b}} & M_l \end{pmatrix} \begin{pmatrix} J^{\tilde{b}}{}_{\tilde{a}} & 0 \\ L^l{}_{\tilde{a}} & K^l{}_k \end{pmatrix} + \begin{pmatrix} \partial_{\tilde{a}} & \partial_k \end{pmatrix} f \,, \tag{4.65}$$

or,

$$A_M = \frac{1}{2} M_N \mathcal{K}^N{}_M + \partial_M f. \tag{4.66}$$

In terms of the field strength derived from (4.66), the Neumann boundary conditions (4.58) can be rewritten as

$$g_{\tilde{a}b} D' X^b = F_{\tilde{a}\tilde{b}} D X^{\tilde{b}} + (F_{\tilde{a}k} + b_{\tilde{a}k}) D X^k, \tag{4.67}$$

i.e. precisely of the general form (2.19). The non-standard boundary term (4.2) can on the other hand be rewritten as

$$\mathcal{S}_{\text{boundary}} = i \int d\tau \, d\theta \, M_N \hat{D} X^N = 2i \int d\tau \, d\theta \, A_N D X^N, \tag{4.68}$$

where the last expression is obtained by using (4.48) and (4.66). This is precisely of the standard form (2.9).

Because of (4.61), both F_{kl} and $\omega_{kl} = g_{km} J^m{}_l$ — which is anti-symmetric because the metric is hermitian with respect to J, but is not closed when $H = db$ is non-trivial — are non-degenerate $(2,0) + (0,2)$ forms with respect to K. This implies that the part of the target space spanned by the twisted chiral superfields is $4l$-dimensional (with $l \in \mathbb{N}$ and $n = 2l$ the number of twisted chiral superfields). We conclude that eqs. (4.42), (4.57) and (4.58) describe a $2(2l+k)$-dimensional brane on a $2(2l+m)$-dimensional target space. Note that when no chiral fields are present — $m = k = 0$ — we recover the maximally coisotropic boundary conditions discussed in [19]. We will therefore henceforth refer to this type of boundary conditions as generalized coisotropic.

For a complete classification of D-branes on bihermitian geometries with two commuting complex structures, it remains to discuss the more general setting where both π_+ and π_- are nonzero. Note however that — since there will be $4l$ π_+-directions — the lowest-dimensional example of such a brane requires a six-dimensional target space, parameterized by twisted chiral fields exclusively. This case was already considered in [19]. An example involving chiral fields as well, will necessarily require a target space of eight dimensions or higher and will thus be physically less relevant. Because in a discussion of the more general case the expressions would become far more complicated, we therefore only briefly outline how more general boundary conditions can be obtained.

To this end, we assume the existence of adapted coordinates $X^{\check{\mu}}$ and $X^{\hat{\mu}}$ (and their complex conjugates), $\check{\mu}, \check{\nu}, \ldots \in \{1, \ldots, l\}$ and $\hat{\mu}, \hat{\nu}, \ldots \in \{l+1, \ldots, n\}$, such that the only non-vanishing components of π_+ and π_- are $\pi_-^{\hat{\mu}}{}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$ and $\pi_+^{\check{\mu}}{}_{\check{\nu}} = \delta^{\check{\mu}}_{\check{\nu}}$. With this, eqs. (4.30) and (4.31) become,

$$\delta X^{\hat{\mu}} = R^{\hat{\mu}}{}_{\bar{\tilde{\nu}}}\,\delta X^{\bar{\tilde{\nu}}} + R^{\hat{\mu}}{}_{\tilde{\beta}}\,\delta X^{\tilde{\beta}} + R^{\hat{\mu}}{}_{\bar{\tilde{\beta}}}\,\delta X^{\bar{\tilde{\beta}}}, \tag{4.69}$$

$$D'X^{\hat{\mu}} = -R^{\hat{\mu}}{}_{\bar{\tilde{\nu}}}\,D'X^{\bar{\tilde{\nu}}} + R^{\hat{\mu}}{}_{\tilde{\beta}}\,DX^{\tilde{\beta}} - R^{\hat{\mu}}{}_{\bar{\tilde{\beta}}}\,DX^{\bar{\tilde{\beta}}}, \tag{4.70}$$

while (4.42) becomes

$$D'X^{\check{k}} = F^{\check{k}}{}_{l}DX^{l} + (F^{\check{k}}{}_{\tilde{a}} + b^{\check{k}}{}_{\tilde{a}})DX^{\tilde{a}}. \tag{4.71}$$

Implementation of this in eq. (4.13) then yields the appropriate boundary conditions on the chiral fields. These will either be of the standard Dirichlet form (4.20) or of a form which will ultimately be equivalent to the Neumann conditions (2.19). One way to find the exact form of these Neumann conditions, is to go to worldvolume coordinates on the brane. These coordinates should be such that eq. (4.69) is trivially satisfied. The field strengths which should enter in these Neumann conditions, can be obtained by using (4.70) and (4.71) in (4.2) and comparing the result to the standard form of the $N = 1$ boundary term (2.9).

In this way a wide variety of branes can be obtained. For example, for an 8-dimensional target space described by one chiral field and three twisted chiral fields, one obtains D3- and D5-branes if $\pi_- = 1$, and D5- and D7-branes when $\pi_- \neq 1$ (note that in this case π_+ cannot equal 1, because this can only happen when there are an even number of twisted chiral fields). The D5- and D7-branes occur when generalized coisotropic Neumann conditions are imposed on two of the twisted chiral fields.

5. Examples

5.1 A four-dimensional target manifold

5.1.1 Generalities

We consider a 4-dimensional target manifold. We can distinguish four different cases.

- It is parameterized in terms of two chiral superfields.
  This case was studied in [19]. It describes either a D0-, or a D2- or a D4-brane on a Kähler manifold wrapping around a holomorphic cycle. It also goes under the name of a B-brane on a Kähler manifold.

- It is parameterized in terms of two twisted chiral superfields.
  This case was also studied in [19]. It describes either a D2-brane wrapped on a lagrangian submanifold of a Kähler manifold or a maximally coisotropic D4-brane on a Kähler manifold. These branes also go under the name of A-branes.

- It is parameterized in terms of one chiral and one twisted chiral superfield.
  This is the case we will study next. As we will show we can either describe D1- or D3-branes on a bihermitian manifold with commuting complex structures.

- It is parameterized in terms of a semi-chiral multiplet.
  This case will be studied elsewhere [20]. However we will briefly touch upon it when discussing duality transformations in section 6. Here it is sufficient to say that this case corresponds to either a D2- or a D4-brane on a bihermitian geometry where $\ker[J_+, J_-] = \emptyset$.

Let us now focus on the case where one has one chiral superfield z and one twisted chiral superfield w. The boundary term eq. (4.13) becomes,

$$\delta\mathcal{S}\Big|_{\text{boundary}} = \int d\tau\, d^2\theta \Big\{\delta\Lambda\big(\bar{\mathbb{D}}' V_z + i\,\bar{\mathbb{D}} W_z\big) - \delta\bar{\Lambda}\big(\mathbb{D}' V_{\bar z} - i\,\mathbb{D} W_{\bar z}\big) + \\ -\delta w\big(V_w - i\,W_w\big) + \delta\bar w\big(V_{\bar w} + i\,W_{\bar w}\big)\Big\}. \qquad (5.1)$$

By choosing appropriate boundary conditions this term should vanish. We have only a single chiral field which leads us immediately to two subcases: either we impose Dirichlet boundary conditions in the chiral direction or not. For the twisted chiral superfield we will always have a Dirichlet and a Neumann boundary condition. So having Dirichlet boundary conditions in the chiral direction will lead to a D1-brane while Neumann boundary conditions in the chiral direction gives a D3-brane. Instead of dwelling on the general case we will focus on a few concrete examples which highlight all subtleties.

5.1.2 D3-branes on T^4

We parametrize T^4 by a chiral z and a twisted chiral w coordinate. For simplicity we choose the torus such that,

$$z \simeq z + \frac{1}{\sqrt{2}}\big(\mathbb{Z} + i\,\mathbb{Z}\big), \qquad w \simeq w + \frac{1}{\sqrt{2}}\big(\mathbb{Z} + i\,\mathbb{Z}\big). \qquad (5.2)$$

Note that one easily generalizes the present analysis to an arbitrary point in the moduli space of T^4. The generalized Kähler potential is simply given by,

$$V = z\bar z - w\bar w. \qquad (5.3)$$

As mentioned here above we can have either D1- or D3-branes. The former is not particularly interesting as by setting z to a constant we end up with a D1-brane wrapping around a lagrangian submanifold of the 2-torus (a trivial concept in two dimensions) parametrized by w which was already discussed in [19]. So we turn to the D3-brane and we choose a simple linear Dirichlet boundary condition,

$$\alpha\, w + \bar\alpha\, \bar w = \beta\, z + \bar\beta\, \bar z, \qquad (5.4)$$

where $\alpha,\, \beta \in \mathbb{C}$ and $\alpha \neq 0$. Consistency of this with eq. (5.2) requires that $\alpha,\, \beta \in \mathbb{Z} + i\,\mathbb{Z}$. In the language of subsection 4.2, this corresponds to taking

$$R^z{}_z = 1, \quad R^w{}_{\bar w} = -\frac{\bar\alpha}{\alpha}, \quad R^w{}_z = \frac{\beta}{\alpha}, \quad R^w{}_{\bar z} = \frac{\bar\beta}{\alpha}. \qquad (5.5)$$

Because of eqs. (3.14) and (3.17), eq. (5.4) immediately implies the Neumann boundary condition,

$$\alpha\, D'w - \bar{\alpha}\, D'\bar{w} = \beta\, Dz - \bar{\beta}\, D\bar{z}. \tag{5.6}$$

Requiring now that the terms proportional to δw and $\delta\bar{w}$ in the boundary term in the variation of the action, eq. (5.1), vanish yields the boundary potential,

$$W = \frac{i}{2}\frac{\alpha}{\bar{\alpha}}\, w^2 - \frac{i}{2}\frac{\bar{\alpha}}{\alpha}\, \bar{w}^2 + f(z,\bar{z}), \tag{5.7}$$

with $f(z,\bar{z})$ an arbitrary[14] real function of z and $\bar{z}$, and two more Neumann boundary conditions,

$$\begin{aligned}
\mathbb{D}'z &= -\frac{\bar{\beta}}{\bar{\alpha}}\,\mathbb{D}w + \frac{\bar{\beta}}{\alpha}\,\mathbb{D}\bar{w} + i\, f_{z\bar{z}}\,\mathbb{D}z,\\
\bar{\mathbb{D}}'\bar{z} &= +\frac{\beta}{\bar{\alpha}}\,\bar{\mathbb{D}}w - \frac{\beta}{\alpha}\,\bar{\mathbb{D}}\bar{w} - i\, f_{z\bar{z}}\,\bar{\mathbb{D}}\bar{z}.
\end{aligned} \tag{5.8}$$

This is indeed of the general form (4.41) for an almost product structure given by (5.5). So we end up with a D3-brane whose position is given by eq. (5.4). Using eq. (5.6) in eq. (4.2) and comparing the result to eq. (2.9) we identify the U(1) gauge fields,

$$\begin{aligned}
A_z &= \frac{1}{2}\left(i\, f_z - \frac{\beta}{\bar{\alpha}}\, w + \frac{\beta}{\alpha}\,\bar{w}\right),\\
A_{\bar{z}} &= \frac{1}{2}\left(-i\, f_{\bar{z}} + \frac{\bar{\beta}}{\bar{\alpha}}\, w - \frac{\bar{\beta}}{\alpha}\,\bar{w}\right),\\
A_w &= A_{\bar{w}} = 0,
\end{aligned} \tag{5.9}$$

as anticipated in eqs. (4.39).

As mentioned in subsection 4.2, one can show that when using worldvolume coordinates, the Neumann boundary conditions (5.6) and (5.8) reduce to a more familiar form. Writing $\alpha = m_1 + i\, m_2$ and $\beta = m_3 + i\, m_4$, where $m_i \in \mathbb{Z}$ and assuming $m_2 \neq 0$, we introduce the worldvolume coordinates r, s and t, and write,

$$z = r + i\, s, \quad w = \left(1 + i\,\frac{m_1}{m_2}\right) t - i\,\frac{m_3}{m_2}\, r + i\,\frac{m_4}{m_2}\, s. \tag{5.10}$$

The non-vanishing components of the pullback of the U(1) fieldstrength are then given by,

$$F_{rs} = -2\,\frac{m_1}{m_2}\,\frac{m_3^2 + m_4^2}{m_1^2 + m_2^2} - \frac{1}{2}(f_{rr} + f_{ss}), \quad F_{rt} = -2\,\frac{m_4}{m_2}, \quad F_{st} = -2\,\frac{m_3}{m_2}. \tag{5.11}$$

Since there is no torsion present in this example, the U(1) field strength F and the invariant two-form $\mathcal{F} = F + b$ are equal to each other. Using the above expressions for the components of F, it is not hard to show that eqs. (5.6) and (5.8) are equivalent to the standard Neumann boundary conditions (2.19).

[14] It is clear that this function should obey appropriate periodicity conditions consistent with the global properties of the torus, i.e. $f\big(z + (m + i\, n)/\sqrt{2}, \bar{z} + (m - i\, n)/\sqrt{2}\big) = f(z,\bar{z}) + h(z) + \bar{h}(\bar{z})$ with $h(z)$ an arbitrary holomorphic function and $m, n \in \mathbb{Z}$.

5.1.3 $S^3 \times S^1$

Wess-Zumino-Witten models are non-trivial but still relatively simple examples of non-linear σ-models with $N \geq (2,2)$ [25]. The simplest case is the WZW-model on $\mathrm{SU}(2) \times \mathrm{U}(1)$ (or the Hopf surface $S^3 \times S^1$) which is the only $N = (2,2)$ WZW-model which can be parameterized without the use of semi-chiral superfields. Parameterizing the WZW group element as,

$$\mathcal{G} = \frac{e^{-i \ln \sqrt{z\bar{z}+w\bar{w}}}}{\sqrt{z\bar{z}+w\bar{w}}} \begin{pmatrix} w & \bar{z} \\ -z & \bar{w} \end{pmatrix}, \tag{5.12}$$

it was found in [26, 27], that the generalized Kähler potential is explicitly given by,

$$\begin{aligned} V &= -\int^{w\bar{w}/z\bar{z}} \frac{dq}{q} \ln\left(1+q\right) + \frac{1}{2}\left(\ln z\,\bar{z}\right)^2 \\ &= +\int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln\left(1+q\right) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2 + \ln(w\bar{w})\ln(z\bar{z}). \end{aligned} \tag{5.13}$$

Note that the equality $V(z,\bar{z},w,\bar{w}) = -V(w,\bar{w},z,\bar{z})$ holds modulo a generalized Kähler transformation. The potential eq. (5.13) correctly encodes the metric and the torsion of the group manifold (see eq. (3.11)). Parameterizing $\mathrm{SU}(2) \times \mathrm{U}(1)$ with Hopf coordinates $z = \cos\psi\, e^{\rho+i\phi_1}$, $w = \sin\psi\, e^{\rho+i\phi_2}$, with ϕ_1, ϕ_2, $\rho \in \mathbb{R} \bmod 2\pi$ and $\psi \in [0, \pi/2]$, we find that z, $w \in (\mathbb{C}^2 \setminus 0)/\Gamma$ where Γ is generated by $(z,w) \to (e^{2\pi}\, z, e^{2\pi}\, w)$ which is precisely the definition of a Hopf surface.

We will use the bulk potential,

$$V = +\int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln\left(1+q\right) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2, \tag{5.14}$$

which differs from eq. (5.13) by a generalized Kähler transformation. In addition we have that global consistency requires invariance under,

$$z \to e^{2\pi n}\, z, \quad w \to e^{2\pi n}\, w, \qquad n \in \mathbb{Z}. \tag{5.15}$$

Under this the generalized Kähler potential transforms as,

$$V \to V - 4\pi\, n \ln(w\bar{w}) - 8\pi^2 n^2, \tag{5.16}$$

which is a generalized Kähler transformation eq. (3.9). In order to restore the invariance the boundary potential should transform as well (see eq. (4.3)),

$$W \to W + 4\pi\, n\, i \ln\left(\frac{w}{\bar{w}}\right), \tag{5.17}$$

which should hold modulo the sum of a holomorphic and an anti-holomorphic function of the chiral fields (see eqs. (4.10-4.11)).

i. D1-branes. We first study D1-branes. We impose the Dirichlet boundary conditions,

$$\begin{aligned} z &= z_0, & \bar{z} &= \bar{z}_0, \\ -i \ln \frac{w}{\bar{w}} &= Q'\big(\ln(z_0\bar{z}_0 + w\bar{w})\big), \end{aligned} \tag{5.18}$$

where we parameterized the boundary potential $W\big(\ln(z_0\bar{z}_0 + w\bar{w})\big)$ as,

$$W(x) = Q(x) - x\,Q'(x), \tag{5.19}$$

where $Q'(x) \equiv \partial_x Q(x)$ and $x \equiv \ln(z_0\bar{z}_0 + w\bar{w})$. Requiring this to be consistent with eq. (5.17) gives,

$$Q(x) = f\left(\sin\left(\frac{x}{2}\right)\right) + \frac{m}{2}\,x^2 + a\,x, \tag{5.20}$$

with $f(y) \in \mathbb{R}$ an arbitrary function and $m,\, a \in \mathbb{R}$. Furthermore requiring that the periodicity of the left hand side of the last equation in (5.18) is correctly reproduced by the right hand side forces us to take $m \in \mathbb{Z}$. One recognizes the integer m as the winding number in the ϕ_2 direction, i.e. going once around the circle parameterized by ρ one winds m times around the circle parameterized by ϕ_2.

ii. D3-branes. We now turn to the D3-brane. We introduce some notation,

$$x \equiv \ln(z\bar{z} + w\bar{w}), \qquad y \equiv -i\,\ln\frac{z}{\bar{z}}, \tag{5.21}$$

and we denote a derivative w.r.t. x by a prime. Parameterizing the boundary potential $W(x, z, \bar{z})$ as,

$$W(x, z, \bar{z}) = Q(x, z, \bar{z}) - x\,Q'(x, z, \bar{z})\,, \tag{5.22}$$

and imposing a single Dirichlet boundary condition,

$$-i \ln \frac{w}{\bar{w}} = Q'(x, z, \bar{z}), \tag{5.23}$$

we get that the terms in eq. (5.1) which are proportional to δw and $\delta\bar{w}$ cancel. Using eq. (5.23) in (5.17) we get that,

$$W\big(x + 4\pi n, e^{2\pi n}z, e^{2\pi n}\bar{z}\big) = W(x, z, \bar{z}) - 4\pi n\,Q'(x, z, \bar{z}), \tag{5.24}$$

should hold. Taking once more a derivative of this with respect to x, we find that $Q''(x, z, \bar{z})$ is periodic,

$$Q''\big(x + 4\pi n, e^{2\pi n}z, e^{2\pi n}\bar{z}\big) = Q''(x, z, \bar{z}). \tag{5.25}$$

In principle we could solve this and integrate it to Q while implementing eq. (5.24). However we will limit ourselves here to a simple choice which satisfies all requirements,

$$Q = \frac{m_1}{2}\,x^2 + m_2\,y\,\big(x - \ln z\bar{z}\big). \tag{5.26}$$

With this the boundary potential becomes,

$$W=-\frac{m_1}{2}\,x^2-m_2\,y\,\ln z\bar{z}, \tag{5.27}$$

and the Dirichlet boundary condition is explicitly,

$$-i\ln\frac{w}{\bar{w}}=m_1\,x+m_2\,y. \tag{5.28}$$

As $-i\ln w/\bar{w}$, x and y are all periodic we get that $m_1, m_2\in\mathbb{Z}$. In the language of subsection 4.2, this corresponds to $R^z{}_z=1$ and,

$$\begin{aligned} R^w{}_{\bar{w}} &= \frac{w}{\bar{w}}\,\frac{z\bar{z}+(1+im_1)w\bar{w}}{z\bar{z}+(1-im_1)w\bar{w}},\\ R^w{}_z &= \frac{w}{z}\,\frac{(m_2+im_1)z\bar{z}+m_2w\bar{w}}{z\bar{z}+(1-im_1)w\bar{w}},\\ R^w{}_{\bar{z}} &= -\frac{w}{\bar{z}}\,\frac{(m_2-im_1)z\bar{z}+m_2w\bar{w}}{z\bar{z}+(1-im_1)w\bar{w}}. \end{aligned} \tag{5.29}$$

The Dirichlet boundary condition implies a Neumann boundary condition as well,

$$\mathbb{D}'\ln w\bar{w}=i\,m_1\,\mathbb{D}x+i\,m_2\,\mathbb{D}y. \tag{5.30}$$

Using eq. (5.28) in the boundary term in the variation of the action, eq. (5.1), one finds that the remaining terms proportional to $\delta\Lambda$ and $\delta\bar{\Lambda}$ vanish provided two more Neumann boundary conditions are imposed,

$$\begin{aligned} \mathbb{D}'z &= i\,m_1\,z\,\mathbb{D}x-\frac{m_2}{\bar{z}}\,\mathbb{D}\big(w\bar{w}\big),\\ \bar{\mathbb{D}}'\bar{z} &= -i\,m_1\,\bar{z}\,\bar{\mathbb{D}}x-\frac{m_2}{z}\,\bar{\mathbb{D}}\big(w\bar{w}\big). \end{aligned} \tag{5.31}$$

These are indeed equivalent to (4.41) for an almost product structure given by (5.29), once the other Neumann condition (5.30) is imposed. Using eq. (5.30) in eq. (4.2) and comparing it to eq. (2.9) leads to a U(1) bundle with potential fields,

$$\begin{aligned} A_w &= A_{\bar{w}}=0,\\ A_z &= -\frac{1}{2}\left(V_z-m_2\,\frac{x}{z}\right),\qquad A_{\bar{z}}=-\frac{1}{2}\left(V_{\bar{z}}-m_2\,\frac{x}{\bar{z}}\right), \end{aligned} \tag{5.32}$$

again in agreement with eq. (4.39). Introducing world volume coordinates ρ, ψ and ϕ such that,

$$z=\cos\psi\,e^{\rho+i\,\phi},\qquad w=\sin\psi\,e^{(1+i\,m_1)\rho+i\,m_2\,\phi}, \tag{5.33}$$

we find that the only non-trivial component of the U(1) bundle pullback to the D3-brane is given by $F_{\rho\psi}=-2\big(\cot\psi+m_2\,\tan\psi\big)$. Combining with the NS-NS 2-form pullback to the D3-brane we obtain the pullback of the invariant 2-form $\mathcal{F}=b+F$,

$$\mathcal{F}_{\rho\phi}=-2\,m_1\,\cos^2\psi,\qquad \mathcal{F}_{\rho\psi}=-2\,m_2\,\tan\psi. \tag{5.34}$$

Given this (and the pullback of the metric to the worldvolume), we expect Neumann boundary conditions of the form (see (2.19)),

$$\begin{aligned}(1+m_1^2\sin^2\psi)\,D'\rho+m_1m_2\sin^2\psi\,D'\phi &= -m_2\tan\psi\,D\psi-m_1\cos^2\psi\,D\phi,\\ (1+m_2^2\tan^2\psi)\,D'\phi+m_1m_2\tan^2\psi\,D'\rho &= m_1D\rho,\\ D'\psi &= m_2\tan\psi\,D\rho,\end{aligned}\tag{5.35}$$

which is indeed equivalent to eqs. (5.30) and (5.31). It would be an instructive exercise to repeat the analysis of [28] for this particular manifest $N=2$ supersymmetric case.

5.2 New space-filling branes on T^6

To illustrate the case $\pi_+=1$ of subsection 4.2, we now discuss the (purely Neumann) boundary conditions for a space filling brane on T^6 parameterized by one chiral superfield z and two twisted chiral superfields w^μ, $\mu\in\{1,2\}$, where we impose generalized coisotropic boundary conditions on the twisted chiral fields. Consider the potential

$$V=z\bar z-w^1\bar w^1-w^2\bar w^2+b_1(z\bar w^1+\bar z w^1)+b_2(z\bar w^2+\bar z w^2),\tag{5.36}$$

with constant $b_\mu\in\mathbb{R}$, $\mu\in\{1,2\}$, so that according to (3.11) the b-field has nonzero components $b_{\mu\bar z}=b_\mu$. By virtue of (4.61) and (4.63) this implies the following relation between the U(1) field strength and the components of the complex structure (4.47)

$$\begin{aligned}F_{\mu\nu}&=iK^{\bar\mu}{}_\nu, & F_{\mu\bar\nu}&=iK^{\bar\mu}{}_{\bar\nu},\\ F_{\mu z}&=iL^{\bar\mu}{}_z, & F_{\mu\bar z}&=iL^{\bar\mu}{}_{\bar z}-b_\mu.\end{aligned}\tag{5.37}$$

From (4.61) it follows that the complex structure $K^k{}_l$ cannot be proportional to $J^k{}_l$. A good choice is $K^1{}_{\bar 2}=1$, $K^2{}_{\bar 1}=-1$, and other unrelated components zero. This corresponds to the choice

$$F_{12}=i,\quad F_{1\bar 2}=0.\tag{5.38}$$

On the other hand, the field strength can be computed from the U(1) potentials (4.66). Assuming a quadratic form of W (so that its second derivatives W_{MN} are constants) and with the above choice of K, this implies a relation between components of L and second derivatives of W. It turns out that we can find a non-trivial[15] solution if $W_{\mu\nu}=W_{\mu\bar\nu}=0$. In that case, we find that we have to satisfy the following relations

$$\begin{aligned}L^{\bar 1}{}_z&=W_{1z}-b_2-iW_{\bar 2z},\\ L^{\bar 2}{}_z&=W_{2z}+b_1+iW_{\bar 1z},\end{aligned}\tag{5.39}$$

while the other components of L are fixed by eq. (4.44) to be

$$L^1{}_z=iL^{\bar 2}{}_z,\quad L^2{}_z=-iL^{\bar 1}{}_z,\tag{5.40}$$

[15] Note that putting L and W to zero results in a rather trivial example in the sense that it corresponds to a 4-dimensional maximally coisotropic system of the kind studied in [19] along with two chiral spectator directions trivially wrapped by the brane.

JHEP10(2008)108

so that L is fully determined by specifying e.g. $L^{\bar{1}}{}_z$ and $L^{\bar{2}}{}_z$. Let us take W_{1z}, W_{2z}, $W_{\bar{1}z}$, $W_{\bar{2}z} \in \mathbb{R}$ and write the left hand side of (5.39) as

$$\begin{aligned} L^{\bar{1}}{}_z &\equiv \alpha = \alpha_1 + i\alpha_2, \\ L^{\bar{2}}{}_z &\equiv \beta = \beta_2 + i\beta_1, \end{aligned} \tag{5.41}$$

with α_j and β_j, $j \in \{1,2\}$ real so that (5.39) is solved by

$$\begin{aligned} \alpha_1 &= W_{1z} - b_2, & \alpha_2 &= -W_{\bar{2}z}, \\ \beta_2 &= W_{2z} + b_1, & \beta_1 &= W_{\bar{1}z}, \end{aligned} \tag{5.42}$$

or

$$\begin{aligned} W = &(\alpha_1 + b_2)(w^1 z + \bar{w}^1 \bar{z}) + \beta_1(w^1 \bar{z} + \bar{w}^1 z) \\ &+(\beta_2 - b_1)(w^2 z + \bar{w}^2 \bar{z}) - \alpha_2(w^2 \bar{z} + \bar{w}^2 z) + f(z,\bar{z}), \end{aligned} \tag{5.43}$$

where $f(z,\bar{z})$ is a real function. With this choice for L and W, the components of the U(1) gauge field become (up to gauge transformations)

$$\begin{aligned} A_1 &= -\frac{1}{2}(\beta_2 - b_1)\bar{z} + \frac{1}{2}(\alpha_2 + ib_2)z - \frac{i}{2}w^2, \\ A_2 &= +\frac{1}{2}(\alpha_1 + b_2)\bar{z} + \frac{1}{2}(\beta_1 - ib_1)z + \frac{i}{2}w^1, \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} A_z = &\frac{1}{2}[\beta(\beta - 2b_1) + \alpha(\alpha + 2b_2)]\bar{z} + \frac{i}{2}f_z \\ &+\frac{1}{2}(2i\alpha_1 - \alpha_2 + ib_2)w^1 + \frac{1}{2}(2i\beta_2 - \beta_1 - ib_1)w^2 \\ &+\frac{1}{2}(\beta_2 + 2i\beta_1 - b_1)\bar{w}^1 - \frac{1}{2}(\alpha_1 + 2i\alpha_2 + b_2)\bar{w}^2, \end{aligned} \tag{5.45}$$

and their complex conjugates. These indeed yield the required components of the invariant field strength (see eq. (5.37))

$$\begin{aligned} F_{1z} &= i\alpha, & F_{1\bar{z}} + b_{1\bar{z}} &= \bar{\beta}, \\ F_{2z} &= i\beta, & F_{2\bar{z}} + b_{2\bar{z}} &= -\bar{\alpha}, \end{aligned} \tag{5.46}$$

while for $F_{z\bar{z}}$ we find

$$F_{z\bar{z}} = -i\alpha_2(\alpha_1 + 2b_2) - i\beta_1(\beta_2 - 2b_1) - iW_{z\bar{z}}, \tag{5.47}$$

which shows that for certain solutions, $F_{z\bar{z}}$ will depend not only on $f(z,\bar{z})$, but also on L and the b-field. Note that the choice $\alpha_1 = -b_2$, $\beta_2 = b_1$ and $\alpha_2 = \beta_1 = 0$, leads to a solution where the boundary potential W is trivial (modulo a term $f(z,\bar{z})$), while L is nontrivial and $F_{1z} = -ib_2$ and $F_{2z} = ib_1$. This situation should be contrasted with the case where the b-field vanishes. In that case W necessarily has to be nontrivial for L to be nontrivial (as can be seen from (5.39)).

JHEP10(2008)108

An explicit construction of this kind of space-filling D6-brane on a more non-trivial target space — the simplest candidate being $S^3 \times S^1 \times T^2$ — should in principle be possible. Furthermore, the solution of this subsection should be dual to a coisotropic D5-brane on T^6 and it should be possible to make this duality explicit by the methods developed in the following section. We leave these matters for further investigation.

6. Duality transformations

6.1 Generalities

T-duality transformations in $N = (2, 2)$ supersymmetric non-linear σ-models correspond to duality transformations which interchange the different types of superfields [2, 26, 29, 9, 30, 31]. The simplest ones are those that allow to exchange a chiral for a twisted chiral superfield and vice-versa when an isometry is present. Gauging the isometry, one imposes — using Lagrange multipliers — that the gauge fields are pure gauge. In this way, integrating over the Lagrange multipliers gives back the original model. However when integrating over the gauge fields (or their potentials which are unconstrained superfields) one obtains the dual model.

Let us briefly review the case without boundaries. As a starting point we take the action,

$$\mathcal{S}_{(1)} = 4 \int d^2\sigma \, d^4\theta \left(- \int^Y dq \, W(q, \ldots) + (z + \bar{z}) \, Y \right), \tag{6.1}$$

where Y is an unconstrained $N = (2, 2)$ superfield, z is either a chiral or a twisted chiral superfield and $\cdots$ stands for other, spectator fields. The equations of motion for Y give,

$$z + \bar{z} = W(Y, \ldots), \tag{6.2}$$

which upon inversion gives,

$$Y = U(z + \bar{z}, \ldots). \tag{6.3}$$

Using this to eliminate Y yields the second order dual action,

$$\mathcal{S}_{\text{dual}} = 4 \int d^2\sigma \, d^4\theta \int^{z+\bar{z}} dq \, U(q, \ldots). \tag{6.4}$$

Take now z and $\bar{z}$ to be chiral superfields and varying them yields,

$$\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- Y = \mathbb{D}_+ \mathbb{D}_- Y = 0, \tag{6.5}$$

which is solved by putting $Y = w + \bar{w}$ with w a twisted chiral superfield. If on the other hand we started off with a field z which was twisted chiral we get upon integrating over z and $\bar{z}$,

$$\bar{\mathbb{D}}_+ \mathbb{D}_- Y = \mathbb{D}_+ \bar{\mathbb{D}}_- Y = 0, \tag{6.6}$$

which is now solved by putting $Y = w+\bar{w}$, with w a chiral superfield. The resulting second order action (which is the action one starts with) is in both cases given by,

$$\mathcal{S} = -4 \int d^2\sigma\, d^4\theta \int^{w+\bar{w}} dq\, W(q,\ldots). \tag{6.7}$$

Let us illustrate this using the previous example, the WZW model on the Hopf surface $S^3 \times S^1$. We will first dualize the twisted chiral field to a chiral one. In order to do this we rewrite the potential eq. (5.14) as,

$$V = -\int^{w\bar{w}} \frac{dq}{q} \ln\big(q + z\bar{z}\big). \tag{6.8}$$

With this the first order action is given by,

$$\mathcal{S}_{(1)} = -4 \int d^2\sigma\, d^2\theta\, d^2\hat{\theta} \left\{ \int^{e^Y} \frac{dq}{q} \ln\big(q + z\bar{z}\big) + Y \ln z'\bar{z}' \right\}, \tag{6.9}$$

where Y is an unconstrained superfield and z' is a chiral superfield. Integrating over $\ln z'$ and $\ln \bar{z}'$ gives the original model back. Integrating over Y gives the dual model with action,

$$\mathcal{S}_{\text{dual}} = -4 \int d^2\sigma\, d^2\theta\, d^2\hat{\theta} \left\{ \int^{z''\bar{z}''} \frac{dq}{q} \ln\big(1-q\big) - \frac{1}{2}\big(\ln z'\bar{z}'\big)^2 \right\}, \tag{6.10}$$

where we performed a change of coordinates $z \to z'' = z\, z'$. Following the duality transformation in detail we find that the chiral fields satisfy $|z''| \leq 1$ and $z' \simeq e^{2\pi(n_1 + i n_2)}\, z'$ with $n_1,\, n_2 \in \mathbb{Z}$. So the target manifold of the dual model factorizes as a product of a disk (with a singular metric) and a torus.[16]

We now dualize the chiral field to a twisted chiral field. The first order action is given by,

$$\mathcal{S}_{(1)} = 4 \int d^2\sigma\, d^2\theta\, d^2\hat{\theta} \left\{ \int^{e^Y} \frac{dq}{q} \ln\Big(1 + \frac{q}{w\bar{w}}\Big) - \frac{1}{2}(\ln w\bar{w})^2 - Y \ln w'\bar{w}' \right\}, \tag{6.11}$$

where w' is twisted chiral and Y is an unconstrained superfield. The original model is recovered by integrating over $\ln w'$ and $\ln \bar{w}'$. Integrating over Y gives the dual model,

$$\mathcal{S}_{\text{dual}} = 4 \int d^2\sigma\, d^2\theta\, d^2\hat{\theta} \left\{ \int^{w''\bar{w}''} \frac{dq}{q} \ln\big(1-q\big) - \frac{1}{2}\big(\ln w'''\bar{w}'''\big)^2 \right\}, \tag{6.12}$$

where we performed the following coordinate transformations,

$$w'' = \frac{1}{w'}, \qquad w''' = w\, w'. \tag{6.13}$$

[16] Note however that a proper treatment of this duality transformation requires the presence of a non-trivial dilaton field in the dual model [26]. Indeed, if this were not the case we would have expected that the dual model is hyper-Kähler which it is not.

One finds that $|w''| \leq 1$ and $w''' \simeq e^{2\pi(n_1+in_2)}w'''$ and the dual model once more factorizes as $D \times T^2$.

In the next section we will extend this duality to the case in which boundaries are present. As already discussed in [19], the main difficulty is the construction of the right boundary terms such that the boundary conditions of the various fields remain consistent with the duality transformation. A crucial partial integration identity is here,

$$\int d^2\sigma\, d^2\theta\, D'\hat{D}'\left(i\,u\,\bar{\mathbb{D}}\bar{\mathbb{D}}'Y + i\,\bar{u}\,\mathbb{D}\mathbb{D}'Y\right) - i\int d\tau\, d^2\theta\left(\bar{\mathbb{D}}'u\,\bar{\mathbb{D}}'Y - \mathbb{D}'\bar{u}\,\mathbb{D}'Y\right)$$
$$= \int d^2\sigma\, d^2\theta\, D'\hat{D}'\; Y\big(z+\bar{z}\big), \tag{6.14}$$

where Y is a real and u ($\bar{u} = u^\dagger$) a complex unconstrained superfield. We also introduced the chiral field $z \equiv i\,\bar{\mathbb{D}}\bar{\mathbb{D}}'u$ and $\bar{z} \equiv i\,\mathbb{D}\mathbb{D}'\bar{u}$. Another essential equation is,

$$\int d^2\sigma\, d^2\theta\, D'\hat{D}'\left(i\,u\,\bar{\mathbb{D}}_+\mathbb{D}_-Y + i\,\bar{u}\,\mathbb{D}_+\bar{\mathbb{D}}_-Y\right) - i\int d\tau\, d^2\theta\left(u\,\bar{\mathbb{D}}_+\mathbb{D}_-Y - \bar{u}\,\mathbb{D}_+\bar{\mathbb{D}}_-Y\right)$$
$$= \int d^2\sigma\, d^2\theta\, D'\hat{D}'\; Y\big(w+\bar{w}\big) - \int d\tau\, d^2\theta\, Y\left(w-\bar{w}\right), \tag{6.15}$$

where Y is a real and u ($\bar{u} = u^\dagger$) a complex unconstrained superfield and where we introduced the twisted chiral field $w \equiv i\,\bar{\mathbb{D}}_+\mathbb{D}_-u$ and $\bar{w} \equiv i\,\mathbb{D}_+\bar{\mathbb{D}}_-\bar{u}$.

Throughout the remainder we will focus on the non-linear σ-model on T^4 and $S^3 \times S^1$ which we believe is sufficient to cover all subtleties involved. Additional examples can easily be constructed.

6.2 Dualizing D3-branes on T^4

6.2.1 Dualizing a chiral field

We start from the example developed in section 5.1.2 where we will dualize the chiral field to a twisted chiral field. Without altering the boundary potential — as the required generalized Kähler transformation yields a total derivative contribution to the boundary term — we take for the Kähler potential $V = -w\bar{w} + (z+\bar{z})^2/2$. Furthermore we assume that the arbitrary function f in the boundary potential eq. (5.7) exhibits the isometry as well: $f = f(z+\bar{z})$. Our starting point is the first order action,

$$\mathcal{S}_{(1)} = \int d^2\sigma\, d^2\theta\, D'\hat{D}'\left(w\bar{w} - \frac{1}{2}Y^2 + i\,u\,\bar{\mathbb{D}}_+\mathbb{D}_-Y + i\,\bar{u}\,\mathbb{D}_+\bar{\mathbb{D}}_-Y\right)$$
$$+i\int d\tau\, d^2\theta\left(\frac{i}{2}\frac{\alpha}{\bar{\alpha}}\,w^2 - \frac{i}{2}\frac{\bar{\alpha}}{\alpha}\,\bar{w}^2 + f(Y) - u\,\bar{\mathbb{D}}_+\mathbb{D}_-Y + \bar{u}\,\mathbb{D}_+\bar{\mathbb{D}}_-Y\right), \tag{6.16}$$

with $Y \in \mathbb{R}$ and $u \in \mathbb{C}$ unconstrained superfields. From eqs. (5.4)–(5.8) we obtain the boundary conditions,

$$\mathbb{D}\left(\alpha w + \bar{\alpha}\bar{w} - \beta\, Y\right) = 0,$$
$$\bar{\mathbb{D}}\left(\alpha w + \bar{\alpha}\bar{w} - \bar{\beta}\, Y\right) = 0, \tag{6.17}$$

and,

$$\mathbb{D}'Y = \mathbb{D}\left(-\frac{\bar{\beta}}{\bar{\alpha}}\,w + \frac{\bar{\beta}}{\alpha}\,\bar{w} + i\,f'(Y)\right),$$
$$\bar{\mathbb{D}}'Y = \bar{\mathbb{D}}\left(+\frac{\beta}{\bar{\alpha}}\,w - \frac{\beta}{\alpha}\,\bar{w} - i\,f'(Y)\right), \tag{6.18}$$

where $f'(Y) = df/dY$. Integrating over u and $\bar{u}$ in eq. (6.16) yields the original model back. Integrating the first order action eq. (6.16) by parts using eq. (6.15) gives,

$$\mathcal{S}_{(1)} = \int d^2\sigma\, d^2\theta\, D'\hat{D}'\,\left(w\bar{w} - \frac{1}{2}Y^2 + \left(w' + \bar{w}'\right)Y\right)$$
$$+i\int d\tau\, d^2\theta\,\left(\frac{i}{2}\frac{\alpha}{\bar{\alpha}}\,w^2 - \frac{i}{2}\frac{\bar{\alpha}}{\alpha}\,\bar{w}^2 + f(Y) + i\left(w' - \bar{w}'\right)Y\right), \tag{6.19}$$

with w' another twisted chiral field. Integrating over Y gives the dual model. The bulk equation of motion for Y is,

$$Y = w' + \bar{w}'. \tag{6.20}$$

The variation of the boundary term requires more care. Before doing this we note that we can distinguish two cases: $\beta = \bar{\beta}$ and $\beta \neq \bar{\beta}$. Indeed when $b \equiv \beta$ is real, eq. (6.17) implies a Dirichlet boundary condition,

$$\alpha w + \bar{\alpha}\bar{w} = b\,Y, \tag{6.21}$$

which is no longer true if $\beta \neq \bar{\beta}$ where $\alpha w + \bar{\alpha}\bar{w} - \beta\,Y$ becomes a complex boundary chiral field. When $\beta = \bar{\beta}$ we have — as can be seen from eq. (5.4) — no Dirichlet boundary condition in the direction in which we dualize so we expect a D2-brane in the dual theory. For $\beta \neq \bar{\beta}$ the Dirichlet boundary condition eq. (5.4) does depend on the direction in which we dualize resulting in a D4-brane in the dual theory. As the dual model describes an A-brane on a 4-dimensional Kähler manifold, the dual D-brane must be a space-filling coisotropic brane.

i. $\boldsymbol{b = \beta = \bar{\beta}}$. When $b \neq 0$, we get that because of eq. (6.21) the variation of Y is related to that of w and $\bar{w}$. Taking the boundary contribution of the variation of w and $\bar{w}$ into account — e.g. using eq. (4.13) — we get that the variation of the boundary term vanishes provided that the Dirichlet boundary condition,

$$w' - \bar{w}' = -\frac{b}{\bar{\alpha}}\,w + \frac{b}{\alpha}\,\bar{w} + i\,f'(Y), \tag{6.22}$$

holds. This is — using eq. (6.20) — indeed equivalent to eq. (6.18).

The dual action becomes,

$$\mathcal{S}_{\text{dual}} = \int d^2\sigma\, d^2\theta\, D'\hat{D}'\,\left(w\bar{w} + w'\bar{w}'\right)$$
$$+i\int d\tau\, d^2\theta\,\left(f(w' + \bar{w}') - \frac{1}{2}\left(w' + \bar{w}'\right)f'(w' + \bar{w}')\right). \tag{6.23}$$

We get two Dirichlet boundary conditions,

$$
\begin{aligned}
\alpha w + \bar{\alpha}\bar{w} &= b\, w' + b\, \bar{w}', \\
w' - \bar{w}' &= -\frac{b}{\bar{\alpha}}\, w + \frac{b}{\alpha}\, \bar{w} + i\, f'(w' + \bar{w}'),
\end{aligned}
\tag{6.24}
$$

where the first one follows from eq. (6.21) and the second one from eq. (6.22). They imply — because of eq. (3.17) — two more Neumann boundary conditions. We end up with a D2-brane (with a flat U(1) bundle) wrapping around a lagrangian submanifold of T^4.

ii. $\boldsymbol{\beta \neq \bar{\beta}}$. When performing the variation of the boundary term, one needs to realize that because of eq. (6.17), $\alpha w + \bar{\alpha}\bar{w} - \bar{\beta} Y$ is a complex chiral field on the boundary. Solving this in terms of unconstrained superfields and appropriately taking the boundary contributions of the bulk variations of w and $\bar{w}$ into account we get that the variation of the boundary term vanishes provided,

$$
\begin{aligned}
&\mathbb{D}\left(+w' - \bar{w}' + \frac{\bar{\beta}}{\bar{\alpha}}\, w - \frac{\bar{\beta}}{\alpha}\, \bar{w} - i\, f'(Y) \right) = 0, \\
&\bar{\mathbb{D}}\left(-w' + \bar{w}' - \frac{\beta}{\bar{\alpha}}\, w + \frac{\beta}{\alpha}\, \bar{w} + i\, f'(Y) \right) = 0.
\end{aligned}
\tag{6.25}
$$

This is indeed consistent with eqs. (6.18) and (6.20).

Summarizing, the dual action is given by,

$$
\begin{aligned}
\mathcal{S}_{\text{dual}} = &\int d^2\sigma\, d^2\theta\, D'\hat{D}'\, \left(w\bar{w} + w'\bar{w}' \right) \\
&+ i \int d\tau\, d^2\theta \left(\frac{i}{2}\frac{\alpha}{\bar{\alpha}}\, w^2 - \frac{i}{2}\frac{\bar{\alpha}}{\alpha}\, \bar{w}^2 + \frac{i}{2}\, w'^2 - \frac{i}{2}\bar{w}'^2 + f(w' + \bar{w}') \right),
\end{aligned}
\tag{6.26}
$$

and the Neumann boundary conditions can be rewritten as,

$$
\begin{aligned}
\hat{D}w &= i\, \frac{\beta + \bar{\beta}}{\beta - \bar{\beta}}\, Dw + i\, \frac{\alpha\bar{\alpha}(1 - i\, f'') - \beta\bar{\beta}}{\alpha(\beta - \bar{\beta})}\, Dw' - i\, \frac{\alpha\bar{\alpha}(1 + i\, f'') + \beta\bar{\beta}}{\alpha(\beta - \bar{\beta})}\, D\bar{w}', \\
\hat{D}w' &= i\, \frac{\alpha\bar{\alpha}(1 + i\, f'') - \beta\bar{\beta}}{\bar{\alpha}(\beta - \bar{\beta})}\, Dw - i\, \frac{\beta + \bar{\beta}}{\beta - \bar{\beta}}\, Dw' + i\, \frac{\alpha\bar{\alpha}(1 + i\, f'') + \beta\bar{\beta}}{\alpha(\beta - \bar{\beta})}\, D\bar{w},
\end{aligned}
\tag{6.27}
$$

together with the complex conjugate of these expressions. We wrote f'' for $\partial^2 f(w' + \bar{w}')/\partial w'^2$. Denoting the twisted chiral fields w,$\bar{w}$, w' and $\bar{w}'$ collectively by w^a, we can rewrite the Neumann boundary conditions as,

$$
\hat{D}w^a = K^a{}_b D w^b,
\tag{6.28}
$$

where consistency with $\hat{D}^2 = -(i/2)\partial/\partial\tau$ requires that K is a complex structure [19]. Indeed one shows that the Nijenhuis-tensor of the complex structure vanishes due to the symmetry $w' + \bar{w}'$ in the function f. Note that K does not anticommute with the original complex structure. So we get here a maximally coisotropic D4-brane. Using eq. (3.19) and comparing the result with eq. (2.19), we obtain the U(1) fieldstrength,

$$\begin{aligned} F_{ww'} &= \frac{\alpha\bar\alpha(1-i\,f'')+\beta\bar\beta}{\bar\alpha(\beta-\bar\beta)}, & F_{w\bar w} &= -\frac{\beta+\bar\beta}{\beta-\bar\beta}, & F_{w\bar w'} &= -\frac{\alpha\bar\alpha(1+i\,f'')-\beta\bar\beta}{\bar\alpha(\beta-\bar\beta)}, \\ F_{w'\bar w} &= -\frac{\alpha\bar\alpha(1-i\,f'')-\beta\bar\beta}{\alpha(\beta-\bar\beta)}, & F_{w'\bar w'} &= \frac{\beta+\bar\beta}{\beta-\bar\beta}, & F_{\bar w\bar w'} &= -\frac{\alpha\bar\alpha(1+i\,f'')+\beta\bar\beta}{\alpha(\beta-\bar\beta)}. \end{aligned} \tag{6.29}$$

This generalizes some of the configurations studied in e.g. [32–34] and [19].

6.2.2 Dualizing a twisted chiral field

With a generalized Kähler transformation, we can make the isometry manifest in the bulk potential eq. (5.3),

$$V = z\bar z - \frac{1}{2}\big(w+\bar w\big)^2, \tag{6.30}$$

which because of eq. (4.3) modifies the boundary potential eq. (5.7) to,

$$W = \frac{i}{2}\,\frac{\alpha+\bar\alpha}{\alpha\bar\alpha}\,\big(\alpha\,w^2-\bar\alpha\,\bar w^2\big)+f(z,\bar z). \tag{6.31}$$

The fact that the boundary potential does not manifestly reflect the isometry — in fact using the boundary condition eq. (5.4) one shows that it is invariant modulo total derivative terms — constitutes the whole subtlety for this particular case. We can rewrite the boundary condition eq. (5.4) as,

$$\big(\alpha+\bar\alpha\big)\mathbb{D}(w+\bar w)+\big(\alpha-\bar\alpha\big)\mathbb{D}'(w+\bar w) = 2\beta\,\mathbb{D}z, \tag{6.32}$$

together with its complex conjugate. Once more we have to distinguish between two cases: either $\alpha\in\mathbb{R}$ or $\alpha\neq\bar\alpha$. In the former case the dual theory describes a D2-brane while for the latter we will obtain a D4-brane. Finally we can also rewrite the Neumann boundary conditions in eq. (5.8) in an invariant way,

$$2\alpha\bar\alpha\,\mathbb{D}'z+\bar\beta\big(\alpha+\bar\alpha\big)\mathbb{D}'(w+\bar w) = -\bar\beta\big(\alpha-\bar\alpha\big)\mathbb{D}(w+\bar w)+2i\alpha\bar\alpha\,f_{z\bar z}\,\mathbb{D}z, \tag{6.33}$$

and its complex conjugate.

i. $\boldsymbol{a\equiv\alpha=\bar\alpha}$. We first rewrite the boundary potential. Writing $z=\bar{\mathbb{D}}\Lambda$ and $\bar z=\mathbb{D}\bar\Lambda$ we get,

$$\mathcal{S}_{\text{boundary}} = i\int d\tau\,d^2\theta\left\{-\frac{i\beta}{a}\,\Lambda\,\bar{\mathbb{D}}'\big(w+\bar w\big)+\frac{i\bar\beta}{a}\,\bar\Lambda\,\mathbb{D}'\big(w+\bar w\big)+f(z,\bar z)\right\}. \tag{6.34}$$

Using this we write the first order action,

$$\begin{aligned} \mathcal{S}_{(1)} = &\int d^2\sigma\,d^2\theta\,D'\hat D'\left\{-z\bar z+\frac{1}{2}Y^2-i\,u\,\bar{\mathbb{D}}\bar{\mathbb{D}}'Y-i\,\bar u\,\mathbb{D}\mathbb{D}'Y\right\} \\ &+i\int d\tau\,d^2\theta\left\{-\frac{i\beta}{a}\,\Lambda\,\bar{\mathbb{D}}'Y+\frac{i\bar\beta}{a}\,\bar\Lambda\,\mathbb{D}'Y+f(z,\bar z)\right\}, \end{aligned} \tag{6.35}$$

where u is an unconstrained complex superfield and Y is a real unconstrained superfield. Varying u yields back the original model. Varying z, $\delta z = \bar{\mathbb{D}}\delta\Lambda$ gives a boundary term which vanishes provided,

$$a\,\mathbb{D}'z + \bar{\beta}\,\mathbb{D}'Y = +i\,a\,f_{z\bar{z}}\,\mathbb{D}z, \tag{6.36}$$

which is consistent with eq. (6.33). Integrating eq. (6.35) by parts,

$$\begin{aligned}\mathcal{S}_{(1)} &= \int d^2\sigma\, d^2\theta\, D'\hat{D}'\left\{-z\bar{z} + \frac{1}{2}Y^2 - Y\left(z' + \bar{z}'\right)\right\}\\ &\quad + i\int d\tau\, d^2\theta\left\{\bar{\mathbb{D}}'Y\left(\frac{i\beta}{a}\Lambda + \bar{\mathbb{D}}'u\right) - \mathbb{D}'Y\left(\frac{i\bar{\beta}}{a}\bar{\Lambda} + \mathbb{D}'\bar{u}\right) + f(z,\bar{z})\right\}.\end{aligned} \tag{6.37}$$

Varying Y gives a bulk equation of motion,

$$Y = z' + \bar{z}', \tag{6.38}$$

and a boundary contribution which vanishes provided,

$$i\beta\,\Lambda + a\,\bar{\mathbb{D}}'u = i\bar{\beta}\bar{\Lambda} + a\,\mathbb{D}'\bar{u} = 0. \tag{6.39}$$

This immediately implies the Dirichlet boundary conditions,

$$z' = \frac{\beta}{a}\,z, \qquad \bar{z}' = \frac{\bar{\beta}}{a}\,\bar{z}, \tag{6.40}$$

which is consistent with eq. (6.32). Combining everything, we get the dual action,

$$\mathcal{S}_{\text{dual}} = \int d^2\sigma\, d^2\theta\, D'\hat{D}'\left(-z\bar{z} - \frac{1}{2}\left(z' + \bar{z}'\right)^2\right) + i\int d\tau\, d^2\theta\, f(z,\bar{z}), \tag{6.41}$$

together with the Dirichlet boundary conditions in eq. (6.40) and the Neumann boundary conditions,

$$a\,\mathbb{D}'z + \bar{\beta}\,\mathbb{D}'z' = ia\,f_{z\bar{z}}\,\mathbb{D}z, \qquad a\,\bar{\mathbb{D}}'\bar{z} + \beta\,\bar{\mathbb{D}}'\bar{z}' = -ia\,f_{z\bar{z}}\,\bar{\mathbb{D}}\bar{z}. \tag{6.42}$$

We see that the dual model describes a B type D2-brane wrapping on a holomorphic cycle determined by eq. (6.40) and we have a non-trivial U(1) bundle with non-vanishing potentials,

$$A_z = +\frac{i}{2}\,f_z, \qquad A_{\bar{z}} = -\frac{i}{2}\,f_{\bar{z}}\,. \tag{6.43}$$

ii. $\boldsymbol{\alpha \neq \bar{\alpha}}$. Using the Dirichlet boundary condition eq. (5.4) we rewrite the boundary potential eq. (6.31) in an invariant way,

$$W = \frac{i\,(\alpha+\bar{\alpha})}{2\,(\alpha-\bar{\alpha})}\left(\frac{1}{\alpha\bar{\alpha}}\left(\beta z + \bar{\beta}\bar{z}\right)^2 - (w+\bar{w})^2\right) + f(z,\bar{z}). \tag{6.44}$$

With this we obtain the first order action,

$$\mathcal{S}_{(1)} = \int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ -z\bar{z} + \frac{1}{2}Y^2 - i\,u\,\bar{\mathbb{D}}\bar{\mathbb{D}}'Y - i\,\bar{u}\,\mathbb{D}\mathbb{D}'Y \right\} \tag{6.45}$$
$$+i\int d\tau\, d^2\theta \left\{ \frac{i\,(\alpha+\bar{\alpha})}{2\,(\alpha-\bar{\alpha})}\left(\frac{1}{\alpha\bar{\alpha}}\left(\beta z+\bar{\beta}\bar{z}\right)^2 - Y^2\right) + f(z,\bar{z}) \right.$$
$$\left. +\bar{\mathbb{D}}'u\left(\bar{\mathbb{D}}'Y + \frac{2\bar{\beta}}{\alpha-\bar{\alpha}}\bar{\mathbb{D}}\bar{z} - \frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\bar{\mathbb{D}}Y\right) - \mathbb{D}'\bar{u}\left(\mathbb{D}'Y - \frac{2\beta}{\alpha-\bar{\alpha}}\mathbb{D}z + \frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\mathbb{D}Y\right)\right\}.$$

Integrating over u and $\bar{u}$ gives us the original model back together with a boundary term which vanishes provided,

$$\bar{\mathbb{D}}'Y + \frac{2\bar{\beta}}{\alpha-\bar{\alpha}}\bar{\mathbb{D}}\bar{z} - \frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\bar{\mathbb{D}}Y = \mathbb{D}'Y - \frac{2\beta}{\alpha-\bar{\alpha}}\mathbb{D}z + \frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\mathbb{D}Y = 0, \tag{6.46}$$

which is consistent with eq. (6.32). Integrating by parts, we rewrite eq. (6.45) as,

$$\mathcal{S}_{(1)} = \int d^2\sigma\, d^2\theta\, D'\hat{D}'\left(-z\bar{z} + \frac{1}{2}Y^2 - Y\left(z'+\bar{z}'\right)\right) \tag{6.47}$$
$$+i\int d\tau\, d^2\theta\left\{\frac{i}{2\alpha\bar{\alpha}}\frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\left(\beta z+\bar{\beta}\bar{z}\right)^2 - i\,\frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\left(\frac{1}{2}Y^2 - Y\left(z'+\bar{z}'\right)\right)\right.$$
$$\left. -i\,\frac{2\bar{\beta}}{\alpha-\bar{\alpha}}\,z'\,\bar{z} - i\,\frac{2\beta}{\alpha-\bar{\alpha}}\,\bar{z}'\,z + f(z,\bar{z})\right\}.$$

Integrating over Y gives both in the bulk and in the boundary,

$$Y = z' + \bar{z}'. \tag{6.48}$$

It is now straightforward to go to the second order expressions. One finds for the bulk potential of the dual model,

$$V = z\bar{z} + z'\bar{z}', \tag{6.49}$$

and for its boundary potential,

$$W = i\,\frac{\beta\bar{\beta}}{\alpha\bar{\alpha}}\,\frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\,z\bar{z} + i\,\frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\,z'\bar{z}' - i\,\frac{2\bar{\beta}}{\alpha-\bar{\alpha}}\,z'\,\bar{z} - i\,\frac{2\beta}{\alpha-\bar{\alpha}}\,\bar{z}'\,z + f(z,\bar{z}). \tag{6.50}$$

The boundary conditions are given by,

$$\mathbb{D}'z = -\frac{\beta\bar{\beta}}{\alpha\bar{\alpha}}\,\frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\,\mathbb{D}z + i\,f_{z\bar{z}}\,\mathbb{D}z + \frac{2\bar{\beta}}{\alpha-\bar{\alpha}}\mathbb{D}z',$$
$$\mathbb{D}'z' = \frac{2\beta}{\alpha-\bar{\alpha}}\,\mathbb{D}z - \frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\mathbb{D}z', \tag{6.51}$$

and their complex conjugates. We end up with a B type D4-brane wrapping around the four torus. Once more we have a non-trivial $\mathrm{U}(1)$ bundle with potentials given by $A_z = i\,W_z/2$, $A_{z'} = i\,W_{z'}/2$ and their complex conjugates.

6.3 Dualizing the branes on $S^3 \times S^1$

6.3.1 Dualizing a twisted chiral field

i. From a D1- to a D2-brane. Our starting point is the D1-brane configuration on $S^3 \times S^1$ discussed in section 5.1.3. For simplicity we choose the Dirichlet boundary conditions for the chiral field as $z = \bar{z} = 0$. We start from the first order action,

$$\begin{aligned} \mathcal{S}_{(1)} = & \int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ \int^{e^Y} \frac{dq}{q} \ln\left(q + z\bar{z}\right) + i\, u\, \bar{\mathbb{D}}\bar{\mathbb{D}}'Y + i\, \bar{u}\, \mathbb{D}\mathbb{D}'Y \right\} \\ & + i \int d\tau\, d^2\theta \left\{ Q(Y) - Y\, Q'(Y) - \bar{\mathbb{D}}'u\big(\bar{\mathbb{D}}'Y + i\, \bar{\mathbb{D}}Q'(Y)\big) + \mathbb{D}'\bar{u}\big(\mathbb{D}'Y - i\, \mathbb{D}Q'(Y)\big) \right\}, \end{aligned} \tag{6.52}$$

where the Lagrange multipliers u and $\bar{u} = u^\dagger$ are unconstrained complex superfields. Varying u and $\bar{u}$ gives the bulk equations of motion $\bar{\mathbb{D}}\bar{\mathbb{D}}'Y = \mathbb{D}\mathbb{D}'Y = 0$ which are solved by putting $Y = \ln w\bar{w}$ with w a twisted chiral superfield. The variation of the Lagrange multipliers yields a boundary term as well which vanishes if Y satisfies the boundary conditions,

$$\begin{aligned} \mathbb{D}'Y &= +i\, Q''(Y)\mathbb{D}Y, \\ \bar{\mathbb{D}}'Y &= -i\, Q''(Y)\bar{\mathbb{D}}Y. \end{aligned} \tag{6.53}$$

Going to the second order action we precisely recover the D1-brane discussed in section 5.1.3 and eq. (6.53) is equivalent to the last boundary condition in eq. (5.18).

Upon integration by parts we rewrite the first order action eq. (6.52) as,

$$\begin{aligned} \mathcal{S}_{(1)} = & \int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ \int^{e^Y} \frac{dq}{q} \ln\left(q + z\bar{z}\right) + Y\, \ln z'\bar{z}' \right\} \\ & + i \int d\tau\, d^2\theta \left\{ Q(Y) - Y\, Q'(Y) - Q'(Y)\, \ln z'\bar{z}' \right\}, \end{aligned} \tag{6.54}$$

where we introduced the chiral field z',

$$\ln z' \equiv i\, \bar{\mathbb{D}}\bar{\mathbb{D}}'u, \qquad \ln \bar{z}' \equiv i\, \mathbb{D}\mathbb{D}'\bar{u}. \tag{6.55}$$

Varying Y in eq. (6.54) gives an explicit expression for Y,

$$Y = \ln\left(1 - z''\bar{z}''\right) - \ln z'\bar{z}'. \tag{6.56}$$

Note that the boundary term in the variation vanishes as well by virtue of eq. (6.56). Going to second order gives the dual model,

$$\begin{aligned} \mathcal{S}_{\text{dual}} = & \int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ \int^{z''\bar{z}''} \frac{dq}{q} \ln\left(1 - q\right) - \frac{1}{2}\left(\ln z'\bar{z}'\right)^2 \right\} \\ & + i \int d\tau\, d^2\theta\, Q(-\ln z'\bar{z}'), \end{aligned} \tag{6.57}$$

Where we redefined $z'' \equiv z\, z'$. We have the boundary conditions,

$$\begin{aligned} z'' &= \bar{z}'' = 0, \\ \mathbb{D}'z' &= +i\, Q''(-\ln z'\bar{z}')\, \mathbb{D}z', \qquad \bar{\mathbb{D}}'\bar{z}' = -i\, Q''(-\ln z'\bar{z}')\, \bar{\mathbb{D}}\bar{z}', \end{aligned} \tag{6.58}$$

where the Neumann boundary conditions follow from combining eqs. (6.53) with (6.56).

The dual model has a target geometry given by $D \times T^2$ with a D2-brane wrapping around the torus.

ii. From a D3- to a D4-brane. We now turn to the dualization of the D3-brane configuration discussed in section 5.1.3. We consider the configuration given by the two Neumann boundary conditions eq. (5.31) for the chiral superfield and the Dirichlet boundary condition and Neumann boundary condition resulting from eq. (5.28) for the twisted chiral superfield. We start from the first order action,

$$\begin{aligned}\mathcal{S}_{(1)} = &\int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ \int^{e^Y} \frac{dq}{q} \ln\left(q + z\bar{z}\right) + i\, u\, \bar{\mathbb{D}}\bar{\mathbb{D}}' Y + i\, \bar{u}\, \mathbb{D}\mathbb{D}' Y \right\} \\ &+ i \int d\tau\, d^2\theta \Big\{ W(Y) - \bar{\mathbb{D}}' u \Big(\bar{\mathbb{D}}' Y + i\, m_1\, \bar{\mathbb{D}} \ln\left(e^Y + z\bar{z}\right) + i\, m_2\, \bar{\mathbb{D}}\, y \Big) \\ &\qquad + \mathbb{D}'\bar{u} \Big(\mathbb{D}' Y - i\, m_1\, \mathbb{D} \ln\left(e^Y + z\bar{z}\right) - i\, m_2\, \mathbb{D}\, y \Big) \Big\}, \end{aligned} \tag{6.59}$$

where $W(Y)$ stands for,

$$W(Y) = -\frac{1}{2} m_1 \big(\ln(z\bar{z} + e^Y) \big)^2 - m_2 y \ln z\bar{z} \tag{6.60}$$

We introduced the Lagrange multipliers u and $\bar{u} = u^\dagger$ as unconstrained complex superfields, just like in the previous section. But now the gauge field Y also has to satisfy the boundary conditions following from eq. (5.31),

$$\begin{aligned} \mathbb{D}' \ln z\bar{z} &= +i\, m_1\, \mathbb{D} \ln\left(e^Y + z\bar{z}\right) - \frac{m_2}{z\bar{z}}\, \mathbb{D}\, e^Y, \\ \bar{\mathbb{D}}' \ln z\bar{z} &= -i\, m_1\, \bar{\mathbb{D}} \ln\left(e^Y + z\bar{z}\right) - \frac{m_2}{z\bar{z}}\, \bar{\mathbb{D}}\, e^Y. \end{aligned} \tag{6.61}$$

Varying u and $\bar{u}$ yields the bulk equations of motion $\bar{\mathbb{D}}\bar{\mathbb{D}}' Y = \mathbb{D}\mathbb{D}' Y = 0$, and a vanishing boundary term if Y satisfies the boundary conditions,

$$\begin{aligned} \mathbb{D}' Y &= +i\, m_1\, \mathbb{D} \ln\left(e^Y + z\bar{z}\right) + i\, m_2\, \mathbb{D}\, y, \\ \bar{\mathbb{D}}' Y &= -i\, m_1\, \bar{\mathbb{D}} \ln\left(e^Y + z\bar{z}\right) - i\, m_2\, \bar{\mathbb{D}}\, y. \end{aligned} \tag{6.62}$$

The bulk equations of motion can be solved by requiring $Y = \ln(w\bar{w})$, where w is a twisted chiral superfield. Implementing this in the first order action eq. (6.59) we recover the original model with the D3-brane from section 5.1.3, for which the boundary conditions eq. (6.62) are equivalent to the Dirichlet condition eq. (5.28) and the boundary conditions eq. (6.61) reduce to eq. (5.31).

Using eq. (6.14) to partially integrate the first order action eq. (6.59), one gets

$$\begin{aligned}\mathcal{S}_{(1)} = &\int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ \int^{e^Y} \frac{dq}{q} \ln\left(q + z\bar{z}\right) + Y\, \ln z'\bar{z}' \right\} \\ &+ i \int d\tau\, d^2\theta\, \Big\{ W(Y) - \big(m_1\, \ln\left(e^Y + z\bar{z}\right) + m_2\, y \big)\, \ln z'\bar{z}' \Big\}, \end{aligned} \tag{6.63}$$

with a new chiral field z',

$$\ln z' \equiv i\,\bar{\mathbb{D}}\bar{\mathbb{D}}' u, \qquad \ln \bar{z}' \equiv i\,\mathbb{D}\mathbb{D}'\bar{u}. \tag{6.64}$$

The equations of motion in the dual picture follow from varying Y in eq. (6.63),

$$Y = \ln\left(1 - z''\bar{z}''\right) - \ln z'\bar{z}', \tag{6.65}$$

in which we introduced $z'' \equiv z\,z'$. Imposing the solution eq. (6.65) also eliminates the boundary term arising from the variation of eq. (6.63) with respect to Y. This solution also allows us to write down the second order action describing the dual model,

$$\begin{aligned}\mathcal{S}_{\text{dual}} = &\int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ \int^{z''\bar{z}''} \frac{dq}{q} \ln\left(1-q\right) - \frac{1}{2}\left(\ln z'\bar{z}'\right)^2 \right\} \\ &+ i \int d\tau\, d^2\theta \left\{ \frac{1}{2} m_1 \left(\ln z'\bar{z}'\right)^2 + i\,m_2 \ln\left(\frac{z''}{\bar{z}''}\right) \ln z''\bar{z}'' - i\,m_2 \ln\left(\frac{z'}{\bar{z}'}\right) \ln z''\bar{z}'' \right\}.\end{aligned} \tag{6.66}$$

Rewriting the boundary conditions eqs. (6.61), (6.62) in terms of the chiral fields z', z'' using eq. (6.65) leads to the following four Neumann boundary conditions,

$$\begin{aligned}\mathbb{D}' \ln\left(1 - z''\bar{z}''\right) &= -m_2\, \mathbb{D} \ln z'\bar{z}', \qquad \bar{\mathbb{D}}' \ln\left(1 - z''\bar{z}''\right) = -m_2\, \bar{\mathbb{D}} \ln z'\bar{z}', \\ \mathbb{D}' \ln z'\bar{z}' &= +i\,m_1\, \mathbb{D} \ln z'\bar{z}' - m_2\, \mathbb{D} \ln\left(\frac{z''}{\bar{z}''}\right), \\ \bar{\mathbb{D}}' \ln z'\bar{z}' &= -i\,m_1\, \bar{\mathbb{D}} \ln z'\bar{z}' + m_2\, \bar{\mathbb{D}} \ln\left(\frac{z''}{\bar{z}''}\right).\end{aligned} \tag{6.67}$$

One can check that these boundary conditions are consistent with eq. (4.13) applied to the dual action in eq. (6.66).

It is clear that the target space geometry of the dual model is described by $D \times T^2$ with a spacefilling D4-brane. The U(1) bundle on the D4-bane can be found by using e.g. eq. (2.9) and eq. (4.2) and leads to the following fieldstrength,

$$\begin{aligned} F_{z'\bar{z}'} &= -i\frac{m_1}{z'\bar{z}'}, \qquad & F_{z''\bar{z}''} &= 0, \\ F_{z'\bar{z}''} &= -\frac{m_2}{z'\bar{z}''}, \qquad & F_{z''\bar{z}'} &= \frac{m_2}{z''\bar{z}'}. \end{aligned} \tag{6.68}$$

6.3.2 Dualizing a chiral field

Our starting point is the D3-brane configuration described in section 5.1.3. The Neumann boundary conditions for the chiral superfield are given by eq. (5.31), while the twisted chiral superfield satisfies the Dirichlet condition eq. (5.28) and the Neumann condition derived from it. We introduce a real unconstrained gauge superfield Y satisfying the boundary condition,

$$\begin{aligned}\mathbb{D}'Y &= i\,m_1\, \mathbb{D} \ln\left(e^Y + w\bar{w}\right) - m_2\, e^{-Y}\, \mathbb{D}\, w\bar{w}, \\ \bar{\mathbb{D}}'Y &= -i\,m_1\, \bar{\mathbb{D}} \ln\left(e^Y + w\bar{w}\right) - m_2 e^{-Y}\, \bar{\mathbb{D}}\, w\bar{w},\end{aligned} \tag{6.69}$$

and,

$$\begin{aligned}\mathbb{D}' \ln(w\bar{w}) &= i\, m_1\, \mathbb{D} \ln\big(e^Y + w\bar{w}\big) + m_2\, \mathbb{D}\, Y,\\ \bar{\mathbb{D}}' \ln(w\bar{w}) &= -i\, m_1\, \bar{\mathbb{D}} \ln\big(e^Y + w\bar{w}\big) + m_2\, \bar{\mathbb{D}}\, Y.\end{aligned} \tag{6.70}$$

This configuration allows us to distinguish two different cases. The first case appears when $m_2 = 0$, which will lead to the dual lagrangian D2-brane. The second situation is characterized by $m_2 \neq 0$, so that we can construct the dual coisotropic D4-brane.

i. From a D3-brane to a lagrangian D2-brane. Let us start by taking $m_2 = 0$. In that case the boundary conditions eqs. (6.69) and (6.70) simplify and we can deduce the following Dirichlet boundary condition from eq. (6.70),

$$-i \ln\left(\frac{w}{\bar{w}}\right) = m_1\, \ln\big(e^Y + w\bar{w}\big)\,. \tag{6.71}$$

We can write the first order action as,

$$\begin{aligned}\mathcal{S}_{(1)} = &- \int d^2\sigma\, d^2\theta\, D'\hat{D}' \bigg\{ \int^{e^Y} \frac{dq}{q} \ln\left(1 + \frac{q}{w\bar{w}}\right) - \frac{1}{2}\big(\ln w\bar{w}\big)^2 \\ &\qquad\qquad - iu\, \bar{\mathbb{D}}_+ \mathbb{D}_- Y - i\bar{u}\, \mathbb{D}_+ \bar{\mathbb{D}}_- Y \bigg\} \\ &+ i \int d\tau\, d^2\theta \bigg\{ -\frac{m_1}{2}\big(\ln\big(w\bar{w} + e^Y\big)\big)^2 - u\, \bar{\mathbb{D}}_+ \mathbb{D}_- Y + \bar{u}\, \mathbb{D}_+ \bar{\mathbb{D}}_- Y \bigg\},\end{aligned} \tag{6.72}$$

again introducing (unconstrained) complex superfields u and $\bar{u} = u^\dagger$. The variation of the action eq. (6.72) with respect to u and $\bar{u}$ yields the equations of motion,

$$\mathbb{D}_- \bar{\mathbb{D}}_+ Y|_{\theta'=\hat{\theta}'=0} = 0 = \mathbb{D}_+ \bar{\mathbb{D}}_- Y|_{\theta'=\hat{\theta}'=0}\,, \tag{6.73}$$

which is solved by $Y = \ln z\bar{z}$ with z a chiral superfield. The second order action gives the original model with a D3-brane, described by the boundary conditions eqs. (5.28), (5.31).

To integrate the action eq. (6.72) by parts we use the identity eq. (6.15) and obtain,

$$\begin{aligned}\mathcal{S}_{(1)} = &- \int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{ \int^{e^Y} \frac{dq}{q} \ln\left(1 + \frac{q}{w\bar{w}}\right) - \frac{1}{2}\big(\ln w\bar{w}\big)^2 - Y \ln s\bar{s} \right\} \\ &+ i \int d\tau\, d^2\theta \bigg\{ -\frac{m_1}{2}\big(\ln\big(w\bar{w} + e^Y\big)\big)^2 + i\, Y\, \ln\left(\frac{s}{\bar{s}}\right) \bigg\},\end{aligned} \tag{6.74}$$

where we introduced the twisted chiral superfield s,

$$\ln s \equiv i\, \bar{\mathbb{D}}_+ \mathbb{D}_- u\,, \qquad \ln \bar{s} \equiv i\, \mathbb{D}_+ \bar{\mathbb{D}}_- \bar{u}\,. \tag{6.75}$$

Varying Y in eq. (6.74) yields the equation of motion,

$$Y = \ln w''\bar{w}'' + \ln\big(1 - w'\bar{w}'\big), \tag{6.76}$$

for which we performed the following coordinate transformation,

$$w' = \frac{1}{s}\,, \qquad w'' = s\, w\,. \tag{6.77}$$

The Dirichlet condition eq. (6.71) implies that a variation of Y at the boundary is related to a variation of w and $\bar{w}$. Therefore, the variation of eq. (6.74) with respect to Y renders a boundary term supplemented with a boundary contribution of the variation of w and $\bar{w}$ — as can be found in e.g. eq. (4.13). By virtue of eq. (6.71) the boundary variation leads to a Dirichlet boundary condition for w',

$$-i\ln\left(\frac{w'}{\bar{w}'}\right) = 0. \tag{6.78}$$

Using eqs. (6.71) and (6.76), we can deduce a (second) Dirichlet boundary condition for w'',

$$-i\,\ln\left(\frac{w''}{\bar{w}''}\right) = m_1\,\ln w''\bar{w}'', \tag{6.79}$$

which is indeed consistent with eq. (6.69) using the equation of motion eq. (6.76).

The dual model is described by the following (second order) action,

$$\begin{aligned}\mathcal{S}_{\text{dual}} &= -\int d^2\sigma\, d^2\theta\, D'\hat{D}'\left\{\int^{w'\bar{w}'}\frac{dt}{t}\ln\left(1-t\right) - \frac{1}{2}\big(\ln w''\bar{w}''\big)^2\right\}\\ &\quad + i\int d\tau\, d^2\theta\left\{-\frac{m_1}{2}\big(\ln w''\bar{w}''\big)^2\right\}.\end{aligned} \tag{6.80}$$

As a check one verifies that the two Dirichlet boundary conditions eqs. (6.78) and (6.79) guarantee that the boundary term in the variation of the action eq. (6.80) vanishes.

The dual target space geometry corresponds to $D\times T^2$ with a D2-brane wrapping along one direction in D and one direction in T^2. On T^2 the brane can only wrap in specific (quantized) directions, given by the integer m_1.

ii. From a D3-brane to a coisotropic D4-brane. To arrive at a coisotropic D4-brane it is necessary to assume $m_2 \neq 0$, and that in this case the gauge superfield Y satisfies the complete boundary conditions eqs. (6.69) and (6.70). The first order action now reads,

$$\begin{aligned}\mathcal{S}_{(1)} &= -\int d^2\sigma\, d^2\theta\, D'\hat{D}'\bigg\{\int^{e^Y}\frac{dq}{q}\ln\left(1+\frac{q}{w\bar{w}}\right) - \frac{1}{2}\big(\ln w\bar{w}\big)^2\\ &\qquad\qquad - i\,u\,\bar{\mathbb{D}}_+\mathbb{D}_-Y - i\,\bar{u}\,\mathbb{D}_+\bar{\mathbb{D}}_-Y\bigg\}\\ &\quad + i\int d\tau\, d^2\theta\bigg\{-\frac{m_1}{2}\left(\ln\left(w\bar{w}+e^Y\right)\right)^2 + Y\left(i\ln\left(\frac{w}{\bar{w}}\right) + m_1\,\ln\left(e^Y + w\bar{w}\right)\right)\\ &\qquad\qquad - u\,\bar{\mathbb{D}}_+\mathbb{D}_-Y + \bar{u}\,\mathbb{D}_+\bar{\mathbb{D}}_-Y\bigg\}.\end{aligned} \tag{6.81}$$

Variation of the unconstrained superfields u, $\bar{u}$ allows us to go back to the original model with a D3-brane, like we mentioned above.

In order to find the coisotropic D4-brane, we need to integrate the action (6.81) by parts using the identity eq. (6.15),

$$\begin{aligned}\mathcal{S}_{(1)} &= -\int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{\int^{e^Y} \frac{dq}{q} \ln\left(1+\frac{q}{w\bar{w}}\right) - \frac{1}{2}\big(\ln w\bar{w}\big)^2 - Y\ln s\bar{s}\right\} \\ &\quad + i\int d\tau\, d^2\theta \Big\{ -\frac{m_1}{2}\left(\ln\left(w\bar{w}+e^Y\right)\right)^2 \\ &\qquad + Y\left(i\ln\left(\frac{w}{\bar{w}}\right) + m_1\ln\left(e^Y+w\bar{w}\right)\right) + i\,Y\ln\left(\frac{s}{\bar{s}}\right)\Big\}.\end{aligned} \tag{6.82}$$

Varying Y in eq. (6.82) gives the same bulk equations of motion eq. (6.76) as above. From the boundary condition eq. (6.70) we notice that Y, w and $\bar{w}$ are constrained at the boundary and need to be solved in terms of unconstrained complex superfields if we want to have the correct boundary variation. One can show that the variation of the boundary terms, including boundary contributions of the bulk variation, vanishes provided,

$$\begin{aligned}\mathbb{D}\left(\ln\left(\frac{w''}{\bar{w}''}\right) - i\,m_1\,\ln w''\bar{w}'' - m_2\ln\left(1-w'\bar{w}'\right)\right) &= 0, \\ \bar{\mathbb{D}}\left(\ln\left(\frac{w''}{\bar{w}''}\right) - i\,m_1\,\ln w''\bar{w}'' + m_2\ln\left(1-w'\bar{w}'\right)\right) &= 0.\end{aligned} \tag{6.83}$$

which is indeed consistent with eqs. (6.76), (6.69) and (6.70). The other two Neumann boundary conditions can then be derived from eqs. (6.70) and (6.76),

$$\begin{aligned}\mathbb{D}\left(\ln\left(\frac{w'}{\bar{w}'}\right) - m_2\,\ln w''\bar{w}''\right) &= 0, \\ \bar{\mathbb{D}}\left(\ln\left(\frac{w'}{\bar{w}'}\right) + m_2\,\ln w''\bar{w}''\right) &= 0.\end{aligned} \tag{6.84}$$

The dual model is given by the second order action,

$$\begin{aligned}\mathcal{S}_{\text{dual}} &= -\int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{\int^{w'\bar{w}'} \frac{dt}{t}\ln\left(1-t\right) - \frac{1}{2}\big(\ln w''\bar{w}''\big)^2\right\} \\ &\quad + i\int d\tau\, d^2\theta \Big\{\frac{m_1}{2}\big(\ln w''\bar{w}''\big)^2 + m_1\ln\left(1-w'\bar{w}'\right)\ln\left(w''\bar{w}''\right) \\ &\qquad + i\,\ln\left(w''\bar{w}''\right)\ln\left(\frac{w''}{\bar{w}''}\right) + i\,\ln\left(1-w'\bar{w}'\right)\ln\left(\frac{w''}{\bar{w}''}\right)\Big\}.\end{aligned} \tag{6.85}$$

One can check the consistency of the dual model by showing that the boundary term eq. (4.13) vanishes. However, the boundary conditions eqs. (6.83) and (6.84) imply that w' and w'' are constrained at the boundary and that the boundary conditions can be solved by introducing chiral boundary superfields. Taking the variation to these boundary fields yields a boundary term which vanishes by virtue of eq. (6.83).

The Neumann boundary conditions eqs. (6.83) and (6.84) may be written as,

$$\begin{aligned}\hat{D}w' &= i\,\frac{m_2^2 w'\bar{w}' - (1-i\,m_1)(1-w'\bar{w}')}{2m_2\,w''\bar{w}'}Dw'' \\ &\quad + i\frac{m_2^2 w'\bar{w}' + (1+i\,m_1)(1-w'\bar{w}')}{2m_2\,\bar{w}''\bar{w}'}D\bar{w}''\,, \\ \hat{D}w'' &= -i\,w''\frac{m_2^2 w'\bar{w}' - (1+i\,m_1)(1-w'\bar{w}')}{2m_2\,w'(1-w'\bar{w}')}Dw' \\ &\quad -i\,w''\frac{m_2^2 w'\bar{w}' + (1+i\,m_1)(1-w'\bar{w}')}{2m_2\,\bar{w}'(1-w'\bar{w}')}D\bar{w}',\end{aligned} \tag{6.86}$$

accompanied by the complex conjugate of these conditions. One can show that the Nijenhuis-tensor of the complex structure indeed vanishes. Hence, the dual model is a coisotropic D4-brane on $D \times T^2$ with the U(1) fieldstrength given by,

$$\begin{aligned} F_{w'w''} &= +\frac{m_2^2 w'\bar{w}' + (1-i\,m_1)(1-w'\bar{w}')}{2m_2\,w'w''(1-w'\bar{w}')}, \quad & F_{w'\bar{w}''} &= +\frac{m_2^2 w'\bar{w}' - (1+i\,m_1)(1-w'\bar{w}')}{2m_2\,w'\bar{w}''(1-w'\bar{w}')}, \\ F_{w''\bar{w}'} &= -\frac{m_2^2 w'\bar{w}' - (1-i\,m_1)(1-w'\bar{w}')}{2m_2\bar{w}'w''(1-w'\bar{w}')}, \quad & F_{\bar{w}'\bar{w}''} &= +\frac{m_2^2 w'\bar{w}' + (1+i\,m_1)(1-w'\bar{w}')}{2m_2\bar{w}'\bar{w}''(1-w'\bar{w}')}\,. \end{aligned} \tag{6.87}$$

This is an interesting example of a maximally coisotropic D-brane as the target manifold $D \times T^2$ is not hyper-Kähler[17] in contrast with previously studied examples of coisotropic branes [32 – 34] and [19].

6.4 Dualizing a chiral/twisted chiral pair to a semi-chiral multiplet

While we will discuss D-branes in a semi-chiral geometry in detail in [20] we can already gain some insights by using the duality transformation proposed in [29] which — if an appropriate isometry is present — allows one to dualize a pair consisting of a chiral and a twisted chiral superfield into a semi-chiral superfield and vice-versa. In [35, 36, 30] and [31], the underlying gauge theory structure has been developed and T-duality transformations were discussed. We first briefly review the case without boundaries closely following the treatment in [30]. Consider a system described by a single chiral superfield z and a single twisted chiral superfield w. The potential has the form,

$$V = V\big(z+\bar{z}, w+\bar{w}, i(z-\bar{z}-w+\bar{w})\big). \tag{6.88}$$

We introduce three unconstrained real superfields Y, $\tilde{Y}$ and $\hat{Y}$ and construct the complex combinations,

$$Y_L \equiv \hat{Y} + i\,\big(Y - \tilde{Y}\big), \qquad Y_R \equiv \hat{Y} + i\,\big(Y + \tilde{Y}\big). \tag{6.89}$$

[17] This can easily be seen from the fact that the Kähler potential does not satisfy the Monge-Ampère equation, $V_{w'\bar{w}'}V_{w''\bar{w}''} - V_{w'\bar{w}''}V_{w''\bar{w}'} = h(w',w'')\bar{h}(\bar{w}',\bar{w}'')$ with h some non-vanishing holomorphic function.

Note that Y_L and Y_R are not independent as $Y_L + \bar{Y}_L = Y_R + \bar{Y}_R$. With this we write down the first order action,

$$\mathcal{S}_{(1)} = 4 \int d^2\sigma\, d^2\theta\, d^2\hat{\theta} \Big\{ V(Y, \tilde{Y}, \hat{Y}) + \upsilon^+ \bar{\mathbb{D}}_+ Y_L + \bar{\upsilon}^+ \mathbb{D}_+ \bar{Y}_L \\ + \upsilon^- \bar{\mathbb{D}}_- Y_R + \bar{\upsilon}^- \mathbb{D}_- \bar{Y}_R \Big\}. \tag{6.90}$$

Integrating over the unconstrained complex fermionic Lagrange multipliers $\upsilon^\pm$ and $\bar{\upsilon}^\pm$ puts the semi-chiral gauge invariant fieldstrengths to zero: $\mathbb{F}_+ = \bar{\mathbb{F}}_+ = \mathbb{F}_- = \bar{\mathbb{F}}_- = 0$ where,

$$\mathbb{F}_+ = i\,\bar{\mathbb{D}}_+ Y_L, \qquad \bar{\mathbb{F}}_+ = i\,\mathbb{D}_+ Y_L, \qquad \mathbb{F}_- = i\,\bar{\mathbb{D}}_- Y_R, \qquad \bar{\mathbb{F}}_- = i\,\mathbb{D}_- Y_R. \tag{6.91}$$

This is solved by putting $Y_L = 2i\,(z - w)$ and $Y_R = 2i\,(z + \bar{w})$ which brings us back to the original model. If on the other hand we integrate over Y, $\tilde{Y}$ and $\hat{Y}$, we obtain the dual model which is now a function of the semi-chiral fields $r \equiv \bar{\mathbb{D}}_+ \upsilon^+$, $\bar{r} \equiv \mathbb{D}_+ \bar{\upsilon}^+$, $s \equiv \bar{\mathbb{D}}_- \upsilon^-$ and $\bar{s} \equiv \mathbb{D}_- \bar{\upsilon}^-$. They satisfy $\bar{\mathbb{D}}_+ r = \mathbb{D}_+ \bar{r} = \bar{\mathbb{D}}_- s = \mathbb{D}_- \bar{s} = 0$ [6].

We now consider boundaries as well. For concreteness we will start from the D3-brane on T^4, discussed in section 5.1.2, as an explicit example. For simplicity we choose $\alpha = i\,a \neq 0$ and $\beta = i\,b$, $a, b \in \mathbb{R}$, which results in a Dirichlet boundary condition of the form,

$$-i\,\big(w - \bar{w}\big) = -i\,\frac{b}{a}\big(z - \bar{z}\big). \tag{6.92}$$

Using a general Kähler transformation we write the potential as,

$$V\big(z + \bar{z}, w + \bar{w}, i(z - \bar{z} - w + \bar{w})\big) = \frac{g+1}{2}\,\big(z + \bar{z}\big)^2 + \frac{g-1}{2}\,\big(w + \bar{w}\big)^2 + \\ \frac{g}{2}\,\big(z - \bar{z} - w + \bar{w}\big)^2, \tag{6.93}$$

where $g \in \mathbb{R}$ and $g \notin \{0, \pm 1\}$. This in its turn implies a boundary potential,

$$W\big(z + \bar{z}, w + \bar{w}, i(z - \bar{z} - w + \bar{w})\big) = i\,g\,\big(w + \bar{w}\big)\big(z - \bar{z} - w + \bar{w}\big), \tag{6.94}$$

where once more we simplified the expressions by taking $f(z, \bar{z}) = 0$ in eq. (5.7). With this we write the first order action,

$$\mathcal{S}_{(1)} = -\int d^2\sigma\, d^2\theta\, D'\hat{D}' \Big(V(Y, \tilde{Y}, \hat{Y}) + \upsilon^+ \bar{\mathbb{D}}_+ Y_L + \bar{\upsilon}^+ \mathbb{D}_+ \bar{Y}_L + \upsilon^- \bar{\mathbb{D}}_- Y_R + \bar{\upsilon}^- \mathbb{D}_- \bar{Y} \Big) \\ + i \int d\tau\, d^2\theta\, \Big(W(Y, \tilde{Y}, \hat{Y}) - i\,\upsilon^+ \bar{\mathbb{D}}_+ Y_L + i\,\bar{\upsilon}^+ \mathbb{D}_+ \bar{Y}_L + i\,\upsilon^- \bar{\mathbb{D}}_- Y_R - i\,\bar{\upsilon}^- \mathbb{D}_- \bar{Y}_R \Big). \tag{6.95}$$

Integrating over the Lagrange multipliers $\upsilon^\pm$ and $\bar{\upsilon}^\pm$ brings us back to the original model. Integrating by parts, we rewrite the first order action as,

$$\mathcal{S}_{(1)} = -\int d^2\sigma\, d^2\theta\, D'\hat{D}' \Big(V(Y, \tilde{Y}, \hat{Y}) + i\,Y\big(r - \bar{r} + s - \bar{s}\big) - i\,\tilde{Y}\big(r - \bar{r} - s + \bar{s}\big) \\ + \hat{Y}\big(r + \bar{r} + s + \bar{s}\big)\Big) + i \int d\tau\, d^2\theta\, \Big(W(Y, \tilde{Y}, \hat{Y}) + Y\big(r + \bar{r} - s - \bar{s}\big) \\ - \tilde{Y}\big(r + \bar{r} + s + \bar{s}\big) - i\,\hat{Y}\big(r - \bar{r} - s + \bar{s}\big)\Big). \tag{6.96}$$

The bulk equations of motion for Y, $\tilde{Y}$ and $\hat{Y}$ give,

$$\begin{aligned} Y &= -\frac{i}{g+1}\big(r-\bar{r}+s-\bar{s}\big),\\ \tilde{Y} &= \frac{i}{g-1}\big(r-\bar{r}-s+\bar{s}\big),\\ \hat{Y} &= \frac{1}{g}\big(r+\bar{r}+s+\bar{s}\big), \end{aligned} \tag{6.97}$$

from which we get the dual potential,

$$\begin{aligned} V_{\text{dual}}(r,\bar{r},s,\bar{s}) = &\frac{1}{2(g+1)}\big(r-\bar{r}+s-\bar{s}\big)^2+\frac{1}{2(g-1)}\big(r-\bar{r}-s+\bar{s}\big)^2\\ &+\frac{1}{2g}\big(r+\bar{r}+s+\bar{s}\big)^2. \end{aligned} \tag{6.98}$$

The treatment of the boundary terms requires more care. Once more we have to distinguish two cases: $a=b$ and $a\neq b$. The former will yield a D2-brane while the latter gives a D4-brane.

i. $\boldsymbol{a=b}$. From eq. (6.92) we find that the gauge fields satisfy a Dirichlet boundary condition,

$$\hat{Y}=0. \tag{6.99}$$

Implementing this in the boundary term of eq. (6.96) we find that integrating over $\tilde{Y}$ and Y yields two Dirichlet boundary conditions in the dual model,

$$r+\bar{r}=s+\bar{s}=0, \tag{6.100}$$

which is consistent with eqs. (6.99) and (6.97). As will be shown in [20], a Dirichlet boundary condition on a semi-chiral superfield is always paired with a Neumann boundary condition, analogous to what happens for a twisted chiral superfield. So we end up with D2-brane whose position is determined by eq. (6.100). The dual generalized Kähler potential is given by eq. (6.98) and the dual boundary potential vanishes, $W_{\text{dual}}(r,\bar{r},s,\bar{s})=0$.

ii. $\boldsymbol{a\neq b}$. In order that our expressions are not unnecessarily cluttered we choose $a=1$ and $b=0$ (other choices yield similar results as long as $a\neq b$). We find now that eq. (6.92) implies the boundary conditions,

$$\bar{\mathbb{D}}\big(\hat{Y}-i\,Y\big)=\mathbb{D}\big(\hat{Y}+i\,Y\big)=0. \tag{6.101}$$

This means that $\hat{Y}-i\,Y$ is a boundary chiral field. Integrating over $\tilde{Y}$ in the boundary term of eq. (6.96) gives an expression equivalent to the last of eq. (6.97). When integrating over Y and $\tilde{Y}$ in the boundary term, we need to take the fact that they are constrained — as expressed by eq. (6.101) – properly into account. We find that the variation vanishes provided the following two Neumann boundary conditions hold,

$$\begin{aligned} &\bar{\mathbb{D}}\big(g\,r+(g-2)\bar{r}-g\,s-(g-2)\bar{s}\big)=0,\\ &\mathbb{D}\big((g-2)r+g\,\bar{r}-(g-2)s-g\,\bar{s}\big)=0. \end{aligned} \tag{6.102}$$

Using eq. (6.97) we can write eq. (6.101) in the second order formalism which yields two more Neumann boundary conditions,

$$\begin{aligned}
&\bar{\mathbb{D}}\big(r + (1+2g)\bar{r} + s + (1+2g)\bar{s}\big) = 0, \\
&\mathbb{D}\big((1+2g)r + \bar{r} + (1+2g)s + \bar{s}\big) = 0.
\end{aligned} \tag{6.103}$$

So now we end up with a D4-brane. Note that the boundary conditions eqs. (6.102) and (6.103) imply the existence of an additional complex structure similar to maximally coisotropic branes on T^4. The generalized Kähler potential is given by eq. (6.98) while the boundary potential is now given by,

$$\begin{aligned}
W_{\mathrm{dual}}(r,\bar{r},s,\bar{s}) = -\frac{i}{g(g+1)}\Big\{&(r+s)\big((1+2g)(r-s) - (\bar{r}-\bar{s})\big) \\
&-(\bar{r}+\bar{s})\big((1+2g)(\bar{r}-\bar{s}) - (r-s)\big)\Big\}.
\end{aligned} \tag{6.104}$$

In fact this particular example already perfectly illustrates the two possible types of boundary conditions one can have when dealing with a semi-chiral multiplet: either one has 2 Dirichlet and 2 Neumann conditions or one has 4 Neumann conditions [20].

7. Conclusions and discussion

We investigated the allowed boundary conditions for a non-linear σ-model in $N = 2$ boundary superspace parameterized by chiral and twisted chiral superfields. This corresponds to classifying D-branes in a bihermitian target manifold geometry for which the two complex structures commute. There is no need to distinguish between A- and B-type superspace boundaries as changing the superspace boundary from B-type (which we used throughout the paper) to A-type amounts to exchanging the chiral superfields for twisted chiral superfields and vice-versa. Having n, $n \in \mathbb{N}$ chiral superfields and $2m_1 + m_2$, $m_1 \in \mathbb{N}$, $m_2 \in \{0, 1\}$, twisted chiral superfields we found that Dp-brane configurations are possible where $p = 2(a+b+m_1)+m_2$ with $a \in \{0, 1, 2, \ldots, n\}$ and $b \in \{0, 1, 2, \ldots, m_1\}$. Whenever $b \neq 0$ one needs an additional complex structure on (a subspace of) the target manifold.

In fact after the initial exploration of semi-chiral fields in the presence of boundaries in section 6.4 we can already anticipate on the results of [20] and illustrate the emerging general picture. In table 1 we summarize the various $N = (2, 2)$ superfields and list their components in $N = 2$ boundary superspace. Chiral fields give rise to constrained boundary superfields while twisted chiral and semi-chiral fields give unconstrained boundary superfields. Looking at the unconstrained boundary superfields one realizes immediately that imposing a Dirichlet boundary condition on them implies a Neumann boundary condition as well. A second type of boundary conditions for the unconstrained boundary superfields is obtained by requiring that a certain combination of them becomes chiral on the boundary. For this one needs an additional complex structure on a subspace of the target manifold. All these cases were illustrated in the examples developed in sections 5 and 6.

In order to make direct contact with string compactifications we have to address the study of D-branes in the six dimensional case. With what we have learned from the previous we find that we can distinguish six different cases according to their superfield content.

JHEP10(2008)108

$N=(2,2)$ fields	$N=(2,2)$ constraints	$N=2$ fields	boundary type
chiral: z, $\bar{z}$	$\bar{\mathbb{D}}_\pm \bar{z} = \mathbb{D}_\pm z = 0$	z, $\mathbb{D}'z$, $\bar{z}$, $\bar{\mathbb{D}}'\bar{z}$	constrained
twisted chiral: w, $\bar{w}$	$\bar{\mathbb{D}}_+ w = \mathbb{D}_- w = 0$, $\mathbb{D}_+ \bar{w} = \bar{\mathbb{D}}_- \bar{w} = 0$	w, $\bar{w}$	unconstrained
semi-chiral: r, $\bar{r}$, s, $\bar{s}$	$\bar{\mathbb{D}}_+ r = \mathbb{D}_+ \bar{r} = 0$, $\bar{\mathbb{D}}_- s = \mathbb{D}_- \bar{s} = 0$	r, $\bar{r}$, s, $\bar{s}$ $\mathbb{D}'r$, $\bar{\mathbb{D}}'\bar{r}$, $\mathbb{D}'s$, $\bar{\mathbb{D}}'\bar{s}$	unconstrained, the last 4 are auxiliary

Table 1: The three types of $N=(2,2)$ superfields together with their reduction to $N=2$ boundary superspace.

1. *3 chiral superfields*
   These are B-branes on a Kähler manifold. We can have D0-, D2-, D4- or D6-branes wrapping on holomorphic cycles.

2. *2 chiral superfields and one twisted chiral superfield*
   We can have D1-, D3- or D5-branes on a bihermitian manifold with commuting complex structures.

3. *1 chiral superfield and two twisted chiral superfields*
   The manifold is bihermitian with commuting complex structures. It allows for D2- or D4-branes with the standard boundary conditions for the twisted chiral superfields. However, if the manifold allows for generalized coisotropic boundary conditions on the twisted chiral superfields one gets in addition a new type of D4-branes and D6-branes.

4. *3 twisted chiral superfields*
   Here we are dealing with A-branes on Kähler manifolds. Either we have a lagrangian D3-brane or a coisotropic D5-brane.

5. *1 chiral superfield and a semi-chiral multiplet*
   The manifold is bihermitian and the kernel of the commutator of the two complex structures is 2-dimensional. If one imposes Dirichlet boundary conditions in the semi-chiral directions one can have D2- or D4-branes. Having full Neumann boundary conditions in the semi-chiral directions gives either D4- or D6-branes.

6. *1 twisted chiral superfield and a semi-chiral multiplet*
   The manifold is bihermitian and the kernel of the commutator of the two complex structures is 2-dimensional. If one imposes Dirichlet boundary conditions in the semi-chiral directions one can have a D3-brane. Having full Neumann boundary conditions in the semi-chiral directions gives a D5-brane.

One sees that even in very simple geometries such as tori — which can be described in terms of any of the field combinations listed above — there is a wealth of D-brane configurations compatible with the $N=2$ supersymmetry. This might have interesting consequences for model building using intersecting brane configurations (see e.g. [34] where the use of coisotropic branes in such settings was initialized).

JHEP10(2008)108

Note from the discussion above that D0- and D1-branes preserving the $N = 2$ supersymmetry are relatively "rare". Indeed D0-branes can only occur on Kähler manifolds solely described in terms of chiral fields. On the other hand we find that $D1$-branes require a target manifold geometry described in terms of a single twisted chiral and an arbitrary number of chiral superfields.

The present analysis clearly motivates a thorough study of semi-chiral superfields in the presence of boundaries as well [20]. One potentially interesting approach could be to "linearize" the model along the lines of [37]. Indeed there it was shown that any model described in terms of m semi-chiral multiplets is equivalent to a gauged σ-model in terms of $2m$ chiral and $2m$ twisted chiral superfields. While the ungauged model has an indefinite metric, we do not see any obvious obstruction to apply the results obtained in this paper to this particular instance.

It is clear that it would be desirable to have a better (global) geometric characterization of these models, e.g. by combining the present results with those in [17] and [16]. Presumably a formulation in terms of generalized complex geometry (see e.g. [23]) will clarify many issues. Indeed, it has been shown [38] that the correct generalization of the notion of A and B branes in this context corresponds to that of a generalized complex submanifold of a generalized Kähler manifold. This is presently under investigation.

The study of the duality transformations between chiral and twisted chiral superfields led to a surprisingly simple method to construct new examples of coisotropic D-branes. In particular we explicitly constructed the first example of a coisotropic D-brane on a manifold which is not hyper-Kähler. The method developed in the examples can easily be extended to a general construction. Take e.g. a model with generalized Kähler potential given by $V(z+\bar{z}, w+\bar{w})$ and a prepotential $Q(z+\bar{z}, w+\bar{w})$. We consider a D3-brane with Dirichlet boundary condition,

$$-i\big(w-\bar{w}\big) = -\frac{Q'}{V''} - ia\left(z-\bar{z}\right), \tag{7.1}$$

where $a \in \mathbb{R}$ and where a prime denotes a derivative with respect to w. The boundary potential W is then given by,

$$W = Q - \frac{Q'V'}{V''}. \tag{7.2}$$

When dualizing the chiral to a twisted chiral field we distinguish two cases:

- $a = 0$ resulting in a dual model where a D2-brane wraps a lagrangian submanifold.
- $a \neq 0$ resulting in a dual model where we have a space filling coisotropic D4-brane.

Another immediate application of the present results would be an analysis of the β-functions for the models discussed. As shown in [39], such a calculation is greatly facilitated by doing it in $N = 2$ superspace which automatically gives the stability conditions that are satisfied by supersymmetric configurations. A particularly simple and straightforward exercise would be the calculation of the 1-loop β-function for the maximally coisotropic

JHEP10(2008)108

D4-brane on T^4 constructed in section 6.2.1 and this would make a direct connection with the results developed in [40].

Finally, the present analysis could perhaps allow to simplify some of the results in [41] by reformulating the gauged linear σ-models in $N = 2$ boundary superspace.

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A. Conventions, notations and identities

We denote the worldsheet coordinates by $\tau \in \mathbb{R}$ and $\sigma \in \mathbb{R}$, $\sigma \geq 0$. Sometimes we use worldsheet light-cone coordinates,

$$\sigma^{\mp\!\!\!\!+} = \tau + \sigma, \qquad \sigma^{=} = \tau - \sigma. \tag{A.1}$$

The $N = (1,1)$ (real) fermionic coordinates are denoted by θ^+ and θ^- and the corresponding derivatives satisfy,

$$D_+^2 = -\frac{i}{2}\,\partial_{\mp\!\!\!\!+}, \qquad D_-^2 = -\frac{i}{2}\,\partial_{=}\,, \qquad \{D_+, D_-\} = 0. \tag{A.2}$$

The $N = (1,1)$ integration measure is explicitely given by,

$$\int d^2\sigma\, d^2\theta = \int d^2\sigma\, D_+ D_-. \tag{A.3}$$

Passing from $N = (1,1)$ to $N = (2,2)$ superspace requires the introduction of two more real fermionic coordinates $\hat\theta^+$ and $\hat\theta^-$ where the corresponding fermionic derivatives satisfy,

$$\hat D_+^2 = -\frac{i}{2}\,\partial_{\mp\!\!\!\!+}, \qquad \hat D_-^2 = -\frac{i}{2}\,\partial_{=}\,, \tag{A.4}$$

and again all other — except for (A.2) — (anti-)commutators do vanish. The $N = (2,2)$ integration measure is,

$$\int d^2\sigma\, d^2\theta\, d^2\hat\theta = \int d^2\sigma\, D_+ D_-\, \hat D_+ \hat D_-. \tag{A.5}$$

Quite often a complex basis is used,

$$\mathbb{D}_\pm \equiv \hat D_\pm + i\, D_\pm, \qquad \bar{\mathbb{D}}_\pm \equiv \hat D_\pm - i\, D_\pm, \tag{A.6}$$

which satisfy,

$$\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = -2i\,\partial_{\mp\!\!\!\!+}, \qquad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = -2i\,\partial_{=}, \tag{A.7}$$

and all other anti-commutators do vanish.

When dealing with boundaries in $N = (2, 2)$ superspace, we introduce various derivatives as linear combinations of the previous ones. We summarize their definitions together with the non-vanishing anti-commutation relations. We have,

$$\begin{aligned} D &\equiv D_+ + D_-, & \hat{D} &\equiv \hat{D}_+ + \hat{D}_-, \\ D' &\equiv D_+ - D_-, & \hat{D}' &\equiv \hat{D}_+ - \hat{D}_-, \end{aligned} \tag{A.8}$$

with,

$$\begin{aligned} D^2 = \hat{D}^2 = D'^2 = \hat{D}'^2 &= -\frac{i}{2}\partial_\tau, \\ \{D, D'\} = \{\hat{D}, \hat{D}'\} &= -i\partial_\sigma. \end{aligned} \tag{A.9}$$

In addition we also use,

$$\begin{aligned} \mathbb{D} &\equiv \mathbb{D}_+ + \mathbb{D}_- = \hat{D} + i\,D, & \mathbb{D}' &\equiv \mathbb{D}_+ - \mathbb{D}_- = \hat{D}' + i\,D', \\ \bar{\mathbb{D}} &\equiv \bar{\mathbb{D}}_+ + \bar{\mathbb{D}}_- = \hat{D} - i\,D, & \bar{\mathbb{D}}' &\equiv \bar{\mathbb{D}}_+ - \bar{\mathbb{D}}_- = \hat{D}' - i\,D'. \end{aligned} \tag{A.10}$$

They satisfy,

$$\begin{aligned} \{\mathbb{D}, \bar{\mathbb{D}}\} = \{\mathbb{D}', \bar{\mathbb{D}}'\} &= -2i\,\partial_\tau, \\ \{\mathbb{D}, \bar{\mathbb{D}}'\} = \{\mathbb{D}', \bar{\mathbb{D}}\} &= -2i\,\partial_\sigma\,. \end{aligned} \tag{A.11}$$

References

[1] L. Alvarez-Gaume and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric Σ model*, *Commun. Math. Phys.* **80** (1981) 443.

[2] S.J. Gates, C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear Σ models*, *Nucl. Phys.* **B 248** (1984) 157.

[3] T.L. Curtright and C.K. Zachos, *Geometry, topology and supersymmetry in nonlinear models*, *Phys. Rev. Lett.* **53** (1984) 1799.

[4] P.S. Howe and G. Sierra, *Two-dimensional supersymmetric nonlinear Σ models with torsion*, *Phys. Lett.* **B 148** (1984) 451.

[5] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler manifolds and off-shell supersymmetry*, *Commun. Math. Phys.* **269** (2007) 833 [hep-th/0512164].

[6] T. Buscher, U. Lindström and M. Roček, *New supersymmetric Σ models with Wess-Zumino terms*, *Phys. Lett.* **B 202** (1988) 94.

[7] I.T. Ivanov, B. Kim and M. Roček, *Complex structures, duality and WZW models in extended superspace*, *Phys. Lett.* **B 343** (1995) 133 [hep-th/9406063].

[8] A. Sevrin and J. Troost, *Off-shell formulation of $N = 2$ non-linear sigma-models*, *Nucl. Phys.* **B 492** (1997) 623 [hep-th/9610102].

[9] J. Bogaerts, A. Sevrin, S. van der Loo and S. Van Gils, *Properties of semi-chiral superfields*, *Nucl. Phys.* **B 562** (1999) 277 [hep-th/9905141].

[10] J. Maes and A. Sevrin, *A note on $N = (2, 2)$ superfields in two dimensions*, *Phys. Lett.* **B 642** (2006) 535 [hep-th/0607119].

[11] H. Ooguri, Y. Oz and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys.* **B 477** (1996) 407 [hep-th/9606112].

[12] A. Hanany and K. Hori, *Branes and $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 513** (1998) 119 [hep-th/9707192].

[13] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry* [hep-th/0005247].

[14] K. Hori, *Linear models of supersymmetric D-branes*, hep-th/0012179.

[15] C. Albertsson, U. Lindström and M. Zabzine, *$N = 1$ supersymmetric sigma model with boundaries. I*, *Commun. Math. Phys.* **233** (2003) 403 [hep-th/0111161].

[16] C. Albertsson, U. Lindström and M. Zabzine, *$N = 1$ supersymmetric sigma model with boundaries. II*, *Nucl. Phys.* **B 678** (2004) 295 [hep-th/0202069].

[17] U. Lindström and M. Zabzine, *$N = 2$ boundary conditions for non-linear sigma models and Landau-Ginzburg models*, *JHEP* **02** (2003) 006 [hep-th/0209098].

[18] P. Koerber, S. Nevens and A. Sevrin, *Supersymmetric non-linear sigma-models with boundaries revisited*, *JHEP* **11** (2003) 066 [hep-th/0309229].

[19] A. Sevrin, W. Staessens and A. Wijns, *The world-sheet description of A and B branes revisited*, *JHEP* **11** (2007) 061 [arXiv:0709.3733].

[20] A. Sevrin, W. Staessens and A. Wijns, *An $N = 2$ worldsheet approach to D-branes in bihermitian geometries: II. Semi-chiral fields*, in preparation.

[21] U. Lindström, M. Rŏcek and P. van Nieuwenhuizen, *Consistent boundary conditions for open strings*, *Nucl. Phys.* **B 662** (2003) 147 [hep-th/0211266].

[22] M. Gualtieri, *Generalized complex geometry*, Ph.D. thesis, Oxford University, Oxford U.K. (2003), math.DG/0401221.

[23] M. Gualtieri, *Generalized complex geometry*, math.DG/0703298.

[24] U. Lindström, M. Rocek, R. von Unge and M. Zabzine, *Generalized Kähler geometry and manifest $N = (2, 2)$ supersymmetric nonlinear sigma-models*, *JHEP* **07** (2005) 067 [hep-th/0411186].

[25] Ph. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Complex structures on parallelized group manifolds and supersymmetric sigma models*, *Phys. Lett.* **B 206** (1988) 71; *Extended supersymmetric sigma models on group manifolds.* 1. *The complex structures*, *Nucl. Phys.* **B 308** (1988) 662.

[26] M. Roček, K. Schoutens and A. Sevrin, *Off-shell WZW models in extended superspace*, *Phys. Lett.* **B 265** (1991) 303.

[27] M. Rocek, C.H. Ahn, K. Schoutens and A. Sevrin, *Superspace WZW models and black holes*, hep-th/9110035.

[28] C. Bachas, M.R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *JHEP* **05** (2000) 048 [hep-th/0003037].

[29] M.T. Grisaru, M. Massar, A. Sevrin and J. Troost, *Some aspects of $N = (2, 2), D = 2$ supersymmetry*, *Fortsch. Phys.* **47** (1999) 301 [hep-th/9801080].

[30] U. Lindström, M. Roček, I. Ryb, R. von Unge and M. Zabzine, *T-duality and generalized Kähler geometry*, arXiv:0707.1696.

[31] W. Merrell and D. Vaman, *T-duality, quotients and generalized Kähler geometry*, arXiv:0707.1697.

[32] A. Kapustin and D. Orlov, *Remarks on A-branes, mirror symmetry, and the Fukaya category*, *J. Geom. Phys.* **48** (2003) 84 [hep-th/0109098].

[33] M. Aldi and E. Zaslow, *Coisotropic branes, noncommutativity, and the mirror correspondence*, *JHEP* **06** (2005) 019 [hep-th/0501247].

[34] A. Font, L.E. Ibanez and F. Marchesano, *Coisotropic D8-branes and model-building*, *JHEP* **09** (2006) 080 [hep-th/0607219].

[35] U. Lindström, M. Roček, I. Ryb, R. von Unge and M. Zabzine, *New* $N = (2,2)$ *vector multiplets*, *JHEP* **08** (2007) 008 [arXiv:0705.3201].

[36] S.J.J. Gates and W. Merrell, $D = 2N = (2,2)$ *semi chiral vector multiplet*, *JHEP* **10** (2007) 035 [arXiv:0705.3207].

[37] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Linearizing generalized Kähler geometry*, *JHEP* **04** (2007) 061, [hep-th/0702126].

[38] M. Zabzine, *Geometry of D-branes for general* $N = (2,2)$ *sigma models*, *Lett. Math. Phys.* **70** (2004) 211 [hep-th/0405240].

[39] S. Nevens, A. Sevrin, W. Troost and A. Wijns, *Derivative corrections to the Born-Infeld action through beta-function calculations in* $N = 2$ *boundary superspace*, *JHEP* **08** (2006) 086 [hep-th/0606255].

[40] A. Kapustin and Y. Li, *Stability conditions for topological D-branes: a worldsheet approach*, hep-th/0311101.

[41] M. Herbst, K. Hori and D. Page, *Phases of* $N = 2$ *theories in* $1 + 1$ *dimensions with boundary*, arXiv:0803.2045.